Linearized Domain Decomposition Approaches for Boundary Value Problems

by

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Abstract

The purpose of this study is to analyze linearized domain decomposition approaches for different nonlinear boundary value problems (BVPs). Nonlinear BVPs frequently form a large system of equations when they are discretized and require parallel computers to solve this system. Domain decomposition approaches are useful to utilize the advantages of parallel computers in order to solve the differential equations. Cherpion's single domain linearized iterative technique is quite useful to solve the nonlinear BVPs that have the form $u^{''} = f(\xi, u, u^{'})$. However with this iterative scheme we are not able to solve the BVP using parallel computers. Therefore we extend this iterative scheme to the domain decomposition context so that we can solve the nonlinear BVP on parallel computers. Theoretical and numerical results are given.

Acknowledgements

I am grateful to my supervisor, Dr. Ronald Haynes for his continuous support, proper guidance and encouragement. I could not have made this journey without his support. Furthermore, I am grateful to the School of Graduate Studies for supporting me financially throughout my degree.

Throughout my academic life I have been in touch with many great teachers; I express my special gratitude to them. I am thankful to my parents and my grandmother for always being there for me. Finally, I thank all my friends for their love and inspiration.

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Chapter 1

Introduction

We are motivated by solving steady state physical problems of the form

$$\mathcal{L}\{u\} = 0 \quad u(0) = a, \quad u(1) = b, \tag{1.1}$$

where $\mathcal{L}$ is a spatial differential operator. If this physical problem has a "difficult" solution, then solving this using a uniform mesh will not give us an accurate result. To solve this problem in a non-uniform physical coordinate x, we transform this physical coordinate to a new computational uniform coordinate ξ, $x = x(\xi)$, where $x(0) = 0, x(1) = 1$ and $\xi \in [0, 1]$. We want to solve the differential equation efficiently for the variable ξ using as few points as possible. Often we wish to use a uniform grid

$$\xi_i = ih, \quad i = 0, 1, ..., N.$$

A standard way to do this is to apply the equidistribution principle. Suppose $M(x, u)$ is the measure of difficulty or error in the solution of the physical problem. We choose $x_i, i = 0, 1, 2, ..., N$, so that

$$\int_{x_{i-1}}^{x_i} M(\tilde{x}, u) d\tilde{x} \equiv \frac{1}{N} \int_0^1 M(\tilde{x}, u) d\tilde{x}. \tag{1.2}$$

The above equation indicates that the error in each subdomain is equal to the average error in the whole domain. Here $\int_0^1 M(\tilde{x},u)d\tilde{x}$ is the total error and $\frac{1}{N}\int_0^1 M(\tilde{x},u)d\tilde{x}$ is the average error. A continuous mesh transformation between the computational coordinate ξ and the physical coordinate x can be achieved by adding (1.2) up to i,

$$\int_0^{x_i} M(\tilde{x},u)d\tilde{x} = \frac{i}{N}\int_0^1 M(\tilde{x},u)d\tilde{x},$$

and by introducing the continuous variable ξ we have

$$\int_0^{x(\xi)} M(\tilde{x},u)d\tilde{x} = \xi\int_0^1 M(\tilde{x},u)d\tilde{x}. \tag{1.3}$$

Differentiating (1.3) twice with respect to ξ, we obtain

$$\frac{d}{d\xi}\left[M(x(\xi),u)\frac{d}{d\xi}x(\xi)\right] = 0. \tag{1.4}$$

The above differential equation is nonlinear with Dirichlet boundary conditions $x(0) = 0$ and $x(1) = 1$. After forming the equation (1.4) we assemble the coupled system of differential equations

$$\mathcal{L}\{u\} = 0 \quad u(0) = a, \quad u(1) = b, \tag{1.5}$$

$$\frac{d}{d\xi}\left\{M(x(\xi),u)\frac{d}{d\xi}x(\xi)\right\} = 0 \quad x(0) = 0, \quad x(1) = 1. \tag{1.6}$$

We can solve this coupled system simultaneously by considering mesh equation (1.6) and physical PDE (1.5) as one large system. This approach is relatively simple and we can solve the coupled system directly by discretizing the whole system and then forming it into one big system of equations. However this approach is not efficient as the simultaneous solution has a highly nonlinear coupling between the physical solution and the mesh. Another disadvantage is that this large system loses the properties that the physical partial differential equation (PDE) and mesh equation may individually have. We can also solve this coupled system through an alternating solution procedure, referred to as the MP procedure [12]. In this iterative approach initially we chose a mesh x^{n-1} to solve the physical equation. With this mesh we solve for an approximate physical solution then with this physical solution we solve the mesh equation for x^n. We will repeat this iteratively until we get the desired level of accuracy in the solution of physical equation. The advantages of this approach are its flexibility and efficiency. In this procedure we can utilize the features of the mesh equation in order to solve the boundary value problem (BVP) (1.6) more efficiently.

This mesh equation can be solved using one of the following algorithms: De Boor's algorithm, an algorithm based on MMPDE5xi or one based on direct optimization of some error bound [12]. However these methods are not suitable for parallel computing. To take advantages of parallel computing, one suitable method of solving this mesh BVP is to use domain decomposition. Domain decomposition (DD) is based on a divide and conquer philosophy. The DD is a technique, which divides a big problem into several subproblems on smaller overlapping or non-overlapping subdomains which form a partition of the original domain. In our analysis we will only consider overlapping subdomains. For overlapping subdomains we can either solve this mesh BVP by the nonlinear domain decomposition method or a linearized domain decomposition method. The main advantage of the nonlinear domain decomposition method is that it provides a fast convergent solution. However, at the same time, the drawback of this method is that in each iteration the solution of many nonlinear systems of equations are required. The linearized domain decomposition method does not have this problem. Motivated by this nonlinear mesh BVP we develop a linearized domain decomposition method which can solve the BVPs of the form $u^{''} = f(\xi, u, u^{'})$, where $f(\xi, u, u^{'})$ depends on $u^{'}$ nonlinearly. The methods developed will be a first step towards a viable linearized domain decomposition method for the mesh BVP.

Linearized methods for nonlinear BVPs have a long history. A linear monotone iterative approach was given by Lui [14] to solve BVPs of the form $u^{''} = f(\xi, u)$ in a convenient way. Cherpion [6] stated and proved another linearized iterative approach on a single domain to solve the BVPs which have the form $u^{''} = f(\xi, u, u^{'})$. We have extended the iteration scheme from [6] to a domain decomposition approach.

Monotonic iterative approaches starting from subsolutions and supersolutions for the BVP of the form (1.7)

$$-u^{''} + f(\xi, u) = 0, \qquad u(a) = u(b) = 0. \tag{1.7}$$

were first introduced by Picard in 1893 [16]. After this work many other monotone iterative methods were developed to solve this type of BVP. In 1931, using subsolutions and supersolutions, G. Scorza Dragoni [7] proved that solutions exist for the BVPs of the form

$$-u^{''} + f(\xi, u, u^{'}) = 0, \qquad u(a) = u(b) = 0. \tag{1.8}$$

He assumed f was continuous and bounded to prove the existence of solution of (1.8). Dragoni modified (1.8) by replacing f by $\bar{f}$, where $\bar{f} = f$ between the subsolution and supersolution. With this choice, he easily controlled the nonlinear dependency of the derivative. He proved that the solution of the modified BVP exists and then using the maximum principle he proved that this solution of the modified BVP is the solution of (1.8). In 1964, Gendzojan [9] developed the monotone iterative methods for the BVP of the form (1.8). This is the very first monotone iterative approach that does not impose any constraint on the nonlinear dependency on the derivative. For a given subsolution α_0 and supersolution β_0 he considered the iteration schemes for $n = 1, 2, 3, ...,$

$$-\alpha_n'' + K(\xi)\alpha_n' + l(\xi)\alpha_n = -f(\xi, \alpha_{n-1}, \alpha_{n-1}') + K(\xi)\alpha_{n-1}' + l(\xi)\alpha_{n-1}, \tag{1.9}$$
$$\alpha_n = 0 \text{ on } \partial\Omega,$$
$$-\beta_n'' + K(\xi)\beta_n' + l(\xi)\beta_n = -f(\xi, \beta_{n-1}, \beta_{n-1}') + K(\xi)\beta_{n-1}' + l(\xi)\beta_{n-1}, \tag{1.10}$$
$$\beta_n = 0 \text{ on } \partial\Omega.$$

Here the functions $K(\xi)$ and $l(\xi)$ depend on the assumptions on the function f. For this reason iteration schemes (1.9) and (1.10) are not feasible from the computational point of view. In 1974, J. Chandra and P.W. Davis [4] developed an iterative method to solve a problem that depends linearly in the derivative. Following that in 1977, S.R. Bernfeld and J. Chandra [1] generalized this method for the problem that depends on the derivative nonlinearly. They considered iterations of the form

$$-\alpha_n'' + M\alpha_n = -f(\xi, \alpha_{n-1}, \alpha_n') + M\alpha_{n-1},$$
$$\alpha_n = 0 \text{ on } \partial\Omega,$$
$$-\beta_n'' + M\beta_n = -f(\xi, \beta_{n-1}, \beta_n') + M\beta_n,$$
$$\beta_n = 0 \text{ on } \partial\Omega,$$

provided that α_0 and β_0 are subsolution and supersolution of BVP (1.8) respectively and M is a constant based on the assumption of f. As α_n' and β_n' appear explicitly on the right side of the scheme, explicit solutions are not possible with these iterates. Thereafter, in 2001 M. Cherpion [6] proposed another iteration scheme which is more simple and computationally feasible. The iteration scheme is given by

$$\alpha_n^{''} - \sqrt[3]{l}K(\xi)\alpha_n^{'} - l\alpha_n = f(\xi, \alpha_{n-1}, \alpha_{n-1}^{'}) - \sqrt[3]{l}K(\xi)\alpha_{n-1}^{'} - l\alpha_{n-1}, \tag{1.11}$$

$$\alpha_n = 0 \text{ on } \partial\Omega,$$

$$\beta_n^{''} - \sqrt[3]{l}K(\xi)\beta_n^{'} - l\beta_n = f(\xi, \beta_{n-1}, \beta_{n-1}^{'}) - \sqrt[3]{l}K(\xi)\beta_{n-1}^{'} - l\beta_{n-1}, \tag{1.12}$$

$$\beta_n = 0 \text{ on } \partial\Omega,$$

where α_0 and β_0 are subsolution and supersolution of BVP (1.8) respectively, $K(\xi)$ is an antisymmetric function on Ω and $l > 0$ is a constant depends on the assumptions of f.

As an example of domain decomposition methods for nonlinear problem the second chapter provides a discussion of the nonlinear domain decomposition method used to solve the moving mesh problem which was published in [8]. The third chapter introduces the existing work [6] of Cherpion's linearized iterative approach on a single domain to solve the BVP of the form $u^{''} = f(\xi, u, u^{'})$. We also present a single domain analysis of the iterations suggested by S.H Lui's DD approach [14] to solve problems of the form $u^{'} = f(\xi, u)$. The fourth chapter contains S.H Lui's DD approach from [14] and a new DD version of Cherpion's iteration is demonstrated and analyzed. Finally the fifth chapter provides the numerical results to support the theory.

Chapter 2

Nonlinear Single Domain and Domain Decomposition methods

To give a flavor of the numerical and parallel approaches to solve boundary value problems we will use BVP (1.6) as a model problem. In this chapter we will discuss nonlinear single domain method and domain decomposition method to solve the mesh BVP (1.6). We will present the finite difference formulation for the nonlinear single domain method.

2.1 Single Domain Approach to solve the BVP

Our focus will be on nonlinear two point boundary value problems. For example the BVP for the mesh maybe written as,

$$\frac{d}{d\xi}\left\{M(x(\xi),u)\frac{d}{d\xi}x(\xi)\right\} = 0 \tag{2.1}$$

$$x(0) = 0, \quad x(1) = 1.$$

Here we demonstrate a typical approach to solve nonlinear BVPs numerically. A standard way to solve (2.1) on $\xi \in [0,1]$ is to use the finite difference approximation. In order to solve this BVP we will discretize using a staggered grid. Let $w(\xi,x) = M(\xi,x)\dfrac{d}{d\xi}x$ then equation (2.1) becomes

$$\frac{dw}{d\xi} = 0.$$

Let x_j approximate $x(\xi_j)$, where $\xi_j = jh, j = 0,1,...,N$. Now we approximate $w(\xi,x)$

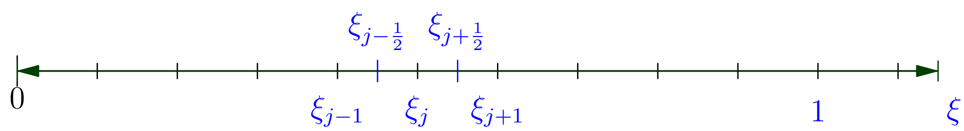


Figure 2.1: Staggered grid discretized.

by taking a short difference at $\xi_{j+\frac{1}{2}}$ and $\xi_{j-\frac{1}{2}}$. Let $w_{j+\frac{1}{2}}$ and $w_{j-\frac{1}{2}}$ be approximations of w at $\xi_{j+\frac{1}{2}}$ and $\xi_{j-\frac{1}{2}}$ respectively, we have

$$w_{j+\frac{1}{2}} \approx M(x_{j-\frac{1}{2}})\frac{x_{j+1} - x_j}{h},$$

and

$$w_{j-\frac{1}{2}} \approx M(x_{j+\frac{1}{2}})\frac{x_j - x_{j-1}}{h}.$$

Hence we get the approximation to (2.1) at ξ_j as

$$\frac{1}{h}\left[w_{j+\frac{1}{2}} - w_{j-\frac{1}{2}}\right] = 0. \tag{2.2}$$

Now we may approximate M at $\xi_{j+\frac{1}{2}}$ and $\xi_{j-\frac{1}{2}}$ using the midpoint formula

$$M(x_{j+\frac{1}{2}}) \approx M\left(\frac{x_{j+1} + x_j}{2}\right),$$

and

$$M(x_{j-\frac{1}{2}}) \approx M\left(\frac{x_j + x_{j-1}}{2}\right).$$

Using these values in equation (2.2), we get the system of discrete algebraic equations

$$M\left(\frac{x_{j+1}+x_j}{2}\right)(x_{j+1}-x_j)-M\left(\frac{x_j+x_{j-1}}{2}\right)(x_j-x_{j-1})=0, \qquad (2.3)$$

$$j=1,2,...,N.$$

$$x_0=0, \qquad x_N=1.$$

Instead of approximating M at $\xi_{j+\frac{1}{2}}$ and $\xi_{j-\frac{1}{2}}$ using midpoint formula we may use the trapezoidal rule

$$M(x_{j+\frac{1}{2}}) \approx \frac{M(x_{j+1})+M(x_j)}{2}$$

and

$$M(x_{j-\frac{1}{2}}) \approx \frac{M(x_j)+M(x_{j-1})}{2}.$$

Then equation (2.2) becomes

$$\left(\frac{M(x_{j+1})+M(x_j)}{2}\right)(x_{j+1}-x_j)-\left(\frac{M(x_j)+M(x_{j-1})}{2}\right)(x_j-x_{j-1})=0, \qquad (2.4)$$

$$j=1,2,...,N,$$

$$x_0=0, \qquad x_N=1.$$

In both cases we have a nonlinear system of equations. We may write these equations as $F(x)=0$ where $F:\mathcal{R}^{N-1}\to\mathcal{R}^{N-1}$ and the j^{th} component of F is given by

$$F_j(x_{j-1},x_j,x_{j+1})=0, \quad j=1,2,...,N.$$

We can solve this nonlinear system using Newton's method. Given an initial guess x^0, the Newton update is given by

$$x^{n+1}=x^n-\left(\frac{\partial F}{\partial x}(x^n)\right)^{-1}F(x^n), \qquad n=0,1,2,... \quad .$$

The Jacobian is given by

$$\frac{\partial F}{\partial x} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & . \quad . \quad . & \frac{\partial F_1}{\partial x_N} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & . \quad . \quad . & \frac{\partial F_2}{\partial x_N} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ \frac{\partial F_N}{\partial x_1} & \frac{\partial F_N}{\partial x_2} & . \quad . \quad . & \frac{\partial F_N}{\partial x_N} \end{bmatrix}.$$

If the initial guess x^0 is sufficiently accurate and the inverse of the Jacobian matrix exists, Newton method will converge quadratically [3], that is error will reduce quadratically which is illustrated in Figure2.7. The order of the local truncation error for (2.4) is $O(\triangle x^2)$ as we have used a second order discretization. This is shown in Figure2.4and2.5.

2.2 Nonlinear Domain Decomposition methods

Nonlinear domain decomposition method is a well known technique to solve differential equations on large scale parallel computers. This method is based on divide-and-conquer philosophy. Basically it splits the problem into several sub problems and solves these sub problems independently by adapting boundary conditions on the interfaces. Here only overlapping subdomains are considered for the nonlinear DD approach. On each subdomain a nonlinear BVP with Dirichlet boundary conditions is solved. In 2012, Haynes and Gander [8] solved the mesh BVP by a parallel nonlinear Schwarz method. Here we will discuss their approaches.

Suppose the domain $\Omega = (0, 1)$ is divided into two subdomains $\Omega_1 = (0, \beta)$ and

$\Omega_2 = (\alpha, 1)$ as in Figure2.2, where α is strictly less than β.

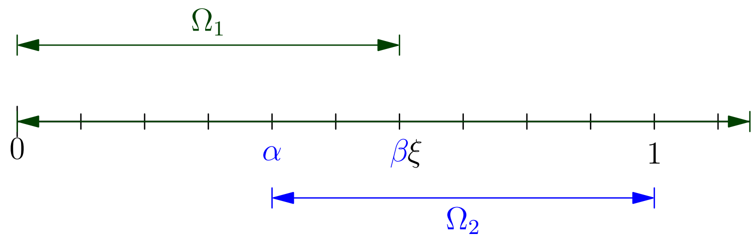


Figure 2.2: Decomposition into two subdomains.

For a given $x_1^0(\alpha)$, $x_2^0(\beta)$, we iterate for $n = 1, 2, 3, ...$

$$\left(M(x_1^n)x_{1,\xi}^n\right)_\xi = 0, \quad \xi \in \Omega_1, \qquad \left(M(x_2^n)x_{2,\xi}^n\right)_\xi = 0, \quad \xi \in \Omega_2, \tag{2.5}$$
$$x_1^n(0) = 0, \quad x_1^n(\beta) = x_2^{n-1}(\beta), \quad x_2^n(\alpha) = x_1^{n-1}(\alpha), \quad x_2^n(1) = 1.$$

If the monitor function $M(x)$ is differentiable and bounded between zero and infinity then the BVP (2.1) has a unique solution [8]. Under these assumptions on the monitor function, iteration (2.5) converges for any initial guesses $x_1^0(\alpha)$, $x_2^0(\beta)$. These statements were proved in [8].

To take the advantages of parallel computer, we assume that the domain is divided into $m > 2$ subdomains that is $\Omega_i = (\alpha_i, \beta_i)$, $i = 1, 2, ..., m$ as shown in Figure2.3.

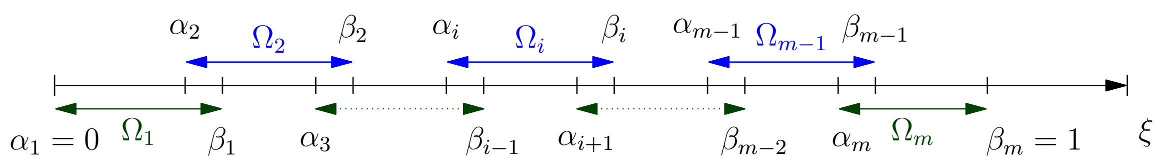


Figure 2.3: Domain decomposed into several subdomains.

Then for given $x_i^0(\alpha_i)$,$x_i^0(\beta_i)$ the iterations are defined for $n = 1, 2, 3, ...$ as

$$\left(M(x_i^n)x_{i,\xi}^n\right)_\xi = 0, x_i^n(\alpha_i) = x_{i-1}^{n-1}(\alpha_i), \quad x_i^n(\beta_i) = x_{i+1}^{n-1}(\beta_i), \quad \xi \in \Omega_i, \tag{2.6}$$

where $i = 1, 2, 3, ..., m$. To ensure the nonadjacent subdomains are not overlapped we assume that $\beta_i < \alpha_{i+2}$ for $i = 1, 2, 3, ..., m-2$. Once the problems are solved in these subdomains we can obtain the whole solution by merging these subdomain solutions. Suppose the monitor function is differentiable and bounded between zero and infinity, then the classical Schwarz iteration (2.6) converges globally on an arbitrary number of subdomains [8]. Numerical illustrations of these statements are provided in Section 5.1.2. We will also observe if we increase the overlap, the DD method will give us faster convergence. Moreover we will see that as the number of subdomains increases the DD method converges more slowly. However there exists many physical problems, where overlapped domain decomposition is not possible. In this situation the classical Schwarz method will not provide convergent solution. We can also get a convergent solution for these types of problems with an optimal Schwarz method or optimized Schwarz method. Optimal Schwarz method can provide a convergent solution in two iterations for two subdomains [8]. However optimal conditions for optimal Schwarz method are not cost effective to use. Optimized Schwarz method can used efficiently providing much faster convergence than classical Schwarz.

2.3 Brief Numerical Remarks

2.3.1 Order of discretization

We consider the mesh density function $M(x) = 1 + x^2$ to analyze the numerical results of nonlinear single domain approach. Figure2.4illustrates the order of the local truncation error in the discretization of two point boundary value problem (2.1), where the mesh density function is discretized using midpoint formula.

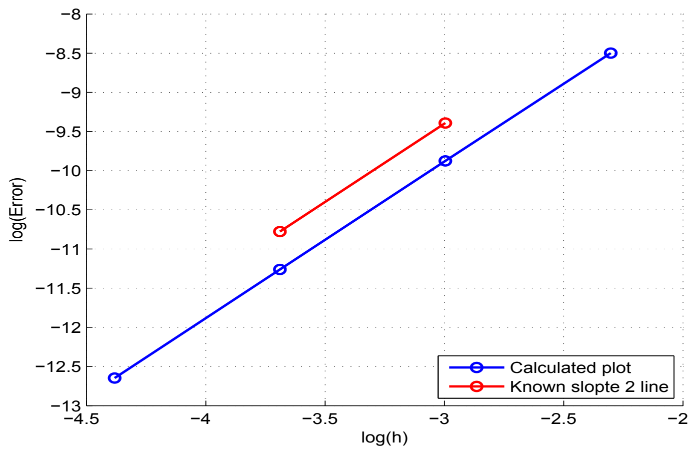


Figure 2.4: Order of the local truncation error using midpoint formula to approximate the mesh density function.

In Figure2.4the blue line shows that the order of the discretization is two, as this line is parallel to a line with known slope of two. Instead of midpoint formula if we use trapezoidal rule to discretize the mesh density function we will also get second order accurate result. Also for each value of h the error is smaller for midpoint than for trapezoidal. From Figure2.5we clearly observe that order of the discretization is two for the trapezoidal rule. Although discretization of mesh density function using midpoint formula and trapezoidal rule both give second order accurate results, the midpoint method provides better results.

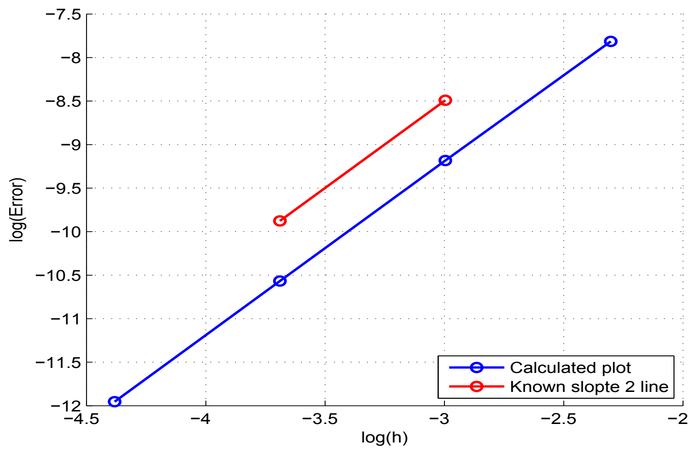


Figure 2.5: Order of the local truncation error using trapezoidal formula to approximate the mesh density function.

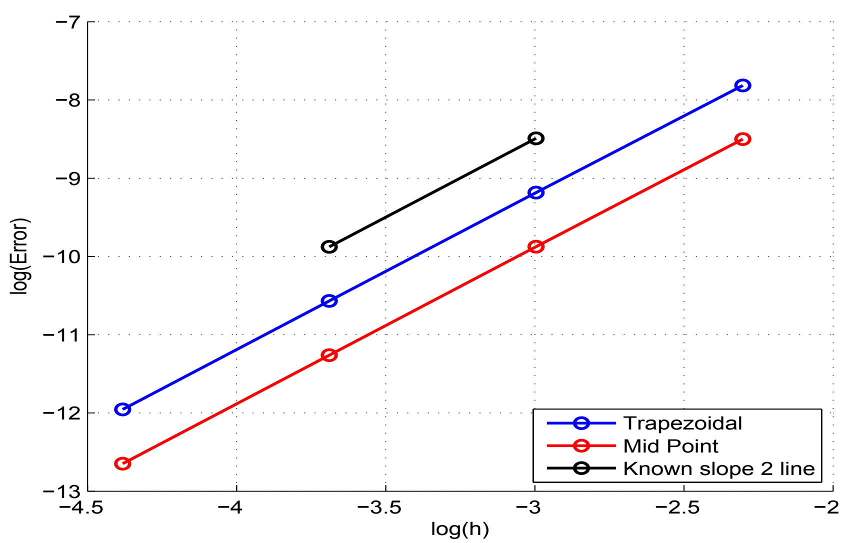


Figure 2.6: Order of discretization both the midpoint formula and the trapezoidal formula.

2.3.2 Rate of Convergence of Newton's method

Theory states that Newton's method will converge quadratically provided that the initial guess is close enough to the true solution and n is large. That is $\epsilon_{n+1} \approx C\epsilon_n^2$ where C is a positive constant and ϵ_n is the error at nth iteration. This implies that if we plot $\log \epsilon_{n+1}$ against $\log \epsilon_n$ for $n = 1, 2, ...$ it will be a slope two line. We certainly observe from Figure2.7that the rate of convergence of Newton's method is two. When the mesh density function is discretized using the midpoint formula.

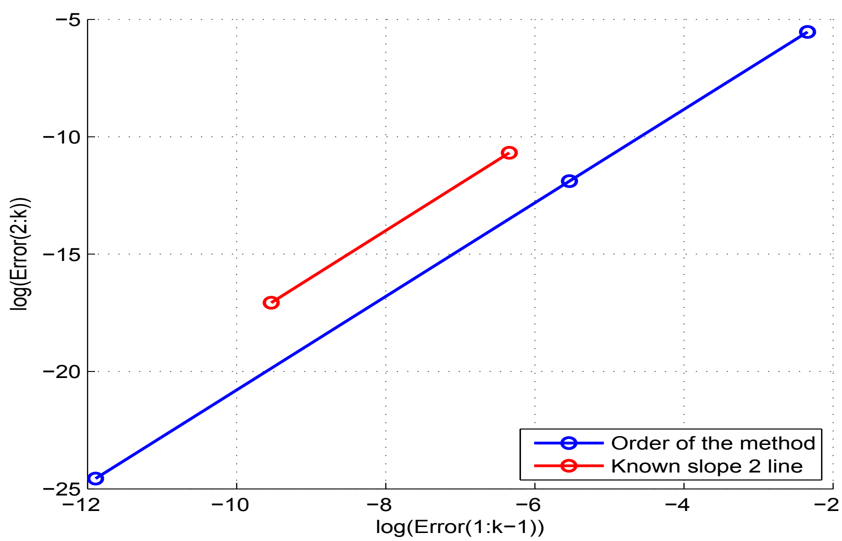


Figure 2.7: Rate of convergence of the Newton's method.

So far we have discussed the nonlinear single domain method and domain decomposition method to solve the BVP. In next chapter we will discuss linearized iterates to solve nonlinear BVPs.

Chapter 3

Linearized Single Domain Iterations

In the previous chapter we have seen that the nonlinear single domain method required a good initial guess and also it required the solution of systems of nonlinear equation in each iteration. However linearized single domain methods do not require the solution of nonlinear systems of equations in each iteration. Furthermore the initial guess does not have to be close to the true solution. In this chapter we will try to modify Lui's linearized DD iteration from [14] to a single domain and elaborately explain Cherpion's linearized single domain iteration from [6] to solve the BVP.

3.1 Linearized single domain method to solve $u'' = f(\xi, u)$

Consider the PDE

$$-\Delta u = f(\xi, u) \text{ on } \Omega, \quad u = h \text{ on } \partial\Omega. \tag{3.1}$$

S.H. Lui stated and proved an iterative approach in [14] to solve this kind of BVP for several subdomains. We will review his domain decomposition analysis in Section 4.1. Here we analyze a similar iteration on a single domain.

We begin with same necessary definitions. The subsolution and supersolution of (3.1) are defined in Definition3.1.1and Definition3.1.2.

Definition 3.1.1 *A function $\underline{u} \in \mathcal{C}^2(\Omega)$ is a subsolution of the PDE (3.1) if*

$$-\Delta\underline{u} - f(\xi,\underline{u}) \leq 0 \text{ on } \Omega \text{ and } \underline{u} \leq h \text{ on } \partial\Omega. \tag{3.2}$$

Definition 3.1.2 *A function $\bar{u} \in \mathcal{C}^2(\Omega)$ is a supersolution of the PDE (3.1) if*

$$-\Delta\bar{u} - f(\xi,\bar{u}) \geq 0 \text{ on } \Omega \text{ and } \bar{u} \geq h \text{ on } \partial\Omega. \tag{3.3}$$

Now we will define a sector of smooth functions which will be needed in the theorem.

Definition 3.1.3 *Suppose $\underline{u}$ is a subsolution and $\bar{u}$ is a supersolution of (3.1) with $\underline{u} \leq \bar{u}$ on Ω. Let $\mathcal{X} = \mathcal{C}^\mu(\bar{\Omega}) \cap \mathcal{C}^2(\Omega)$ for some $0 < \mu < 1$, where $\mathcal{C}^\mu(\bar{\Omega})$ is the space of Hölder continuous functions on $\bar{\Omega}$. Define the sector of smooth functions between $\underline{u}$ and $\bar{u}$ as*

$$\mathcal{A} = \{u \in \mathcal{X} | \underline{u} \leq u \leq \bar{u}\}. \tag{3.4}$$

Furthermore an assumption for the theorem is stated in Assumption1.

Assumption 1 *Assume f is a Hölder continuous function defined on $\bar{\Omega} \times \mathcal{A}$. In addition, suppose there exists some non-negative function $c \in \mathcal{C}^\mu(\bar{\Omega})$ so that for all $\xi \in \Omega$ and $v \leq u \in \mathcal{A}$*

$$-c(\xi)(u - v) \leq f(\xi, u) - f(\xi, v). \tag{3.5}$$

Under this assumption on f for a given subsolution $\underline{u}$ or supersolution $\bar{u}$ with $u^0 = \underline{u}$ or $u^0 = \bar{u}$ we consider the iteration scheme: for $n = 0, 1, 2, ...$

$$-\Delta u^{(n+1)} + cu^{(n+1)} = f(u^{(n)}) + cu^{(n)} \text{ in } \Omega, \tag{3.6}$$
$$u^{(n+1)} = u^{(n)} \text{ on } \partial\Omega.$$

To begin to analyze this iteration we will state some useful lemmas from Pao [15], these are quoted as Lemma3.1.4, Lemma3.1.5and Lemma3.1.7.

Lemma 3.1.4 *Let $\mathcal{L}_p(\Omega)$ be the Banach space with norm $\parallel u \parallel_{\mathcal{L}_p(\Omega)} = \Big(\int_\Omega \mid u(\xi) \mid^p d\xi \Big)^{\frac{1}{p}}$ and $\mathcal{W}_p^2(\Omega)$ be the Sobolev space of all functions in $\mathcal{L}_p(\Omega)$ that have distributional derivatives $\mathcal{D}^l u \in \mathcal{L}_p(\Omega)$ for all $\mid l \mid \leq 2$ with norm $\parallel u \parallel_{\mathcal{W}_p^2(\Omega)} = \Big(\int_\Omega \sum_{|l| \leq 2} \mid D^l u \mid^p \Big)^{\frac{1}{p}}$. If u is solution of (3.6) belongs to $\mathcal{W}_p^2(\Omega)$, then there exists a constant K_1, independent of f and h such that*

$$|| u ||_{\mathcal{W}_p^2(\Omega)} \leq K_1(|| f ||_{\mathcal{L}_p(\Omega)} + || h ||_{2-\frac{1}{p}}).$$

Lemma 3.1.5 *Suppose $\Omega \in \mathcal{R}^N$. For any $p > N$ the Sobolev space $\mathcal{W}_p^2(\Omega)$ is continuously embedded in $\mathcal{C}^{1+\mu}(\bar{\Omega})$ with $\mu = 1 - \frac{N}{p}$. That is there exists a constant K_2 such that for all $u \in \mathcal{W}_p^2(\Omega)$*

$$\mid u \mid_{1+\mu}^{\bar{\Omega}} \leq K_2 \parallel u \parallel_{\mathcal{W}_p^2(\Omega)} .$$

Lemma 3.1.6 *Assume f satisfies (3.5) and also let $c \geq 0$. If the sequence $\{u^{(n)}\}$ defined by (3.6) is bounded in $\mathcal{C}^{1+\mu}(\bar{\Omega})$ then $\{f(\xi, u^n)\}$ is bounded in $\mathcal{C}^\mu(\bar{\Omega})$.*

A Schauder estimate is sufficient enough to show the boundedness for the solution of (3.6). In the following lemma the Schauder estimate is stated from [15].

Lemma 3.1.7 *Let $f(\xi, u) \in \mathcal{C}^{\mu}(\bar{\Omega})$ and $h \in \mathcal{C}^{2+\mu}(\partial\Omega)$. Then the Schauder estimate for the solution $u \in \mathcal{C}^{2+\mu}(\bar{\Omega})$ of (3.6) is*

$$| u |_{2+\mu}^{\bar{\Omega}} \leq K_3 \Big(| f(\xi, u) |_{\mu}^{\bar{\Omega}} + | h |_{1+\mu}^{\partial\Omega} \Big)$$

where K_3 is a constant independent of u,$f(\xi, u)$ and h. Here $| \, . \, |_{\mu}^{\bar{\Omega}}$, $| \, . \, |_{1+\mu}^{\partial\Omega}$ and $| \, . \, |_{2+\mu}^{\bar{\Omega}}$ are the norms defined in spaces $\mathcal{C}^{\mu}(\bar{\Omega})$, $\mathcal{C}^{1+\mu}(\partial\Omega)$ and $\mathcal{C}^{1+\mu}(\bar{\Omega})$ respectively.

The maximum principle is the basic tool to analyze monotone iterations. One useful form of this principle is the weak maximum principle which is stated below.

Lemma 3.1.8 *Let $w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ satisfy,*

$$-\triangle w + cw \geq 0 \text{ in } \Omega, \qquad w \geq 0 \text{ on } \partial\Omega.$$

If $c \geq 0$ then $w \geq 0$ in Ω.

Proof: Suppose w is not positive in Ω. Since w is continuous, differentiable and non-negative on the boundary there must be a minimum negative value of w. Assume this occurs at ξ_0 in the interior of Ω. Since w has a minimum value at ξ_0, we have $\triangle w(\xi_0) \geq 0$. That implies $cw(\xi_0) \geq 0$ which is a contradiction as $c \geq 0$. Hence we conclude that $w \geq 0$ in Ω. ■

The **Arzela-Ascoli** theorem is very useful for proving the convergence of sequences. This theorem is stated as follows. The proof can be found in [11].

Theorem 3.1.9 *Let $\mathcal{X}$ be a compact metric space and let $\{f_n\}$ be a uniformly bounded, equicontinuous sequence of functions on $\mathcal{X}$. Then the sequence $\{f_n\}$ has a uniformly convergent subsequence.*

Here we state another lemma from D. Gilbarg and N.S. Trudinger [10].

Lemma 3.1.10 *Let Ω be a $\mathcal{C}^{q_1+\mu}$ domain in $\mathcal{R}^N$ with $q_1 \geq 1$ and let $\mathcal{D}$ be a bounded set in $\mathcal{C}^{q_1+\mu}(\bar{\Omega})$. Then $\mathcal{D}$ is precompact in $\mathcal{C}^{q_2+\theta}(\bar{\Omega})$ if $q_2 + \theta < q_1 + \mu$.*

Remark 1 *Lemma3.1.10says if we take a sequence from a set $\mathcal{D}$, where $\mathcal{D}$ is bounded set in $\mathcal{C}^{q_1+\mu}(\bar{\Omega})$, then this sequence has a convergent subsequence in $\mathcal{C}^{q_2+\theta}(\bar{\Omega})$ with $q_2 + \theta < q_1 + \mu$.*

The following theorem is a modified version of S.H. Lui's [14] linearized DD approach to single domain.

Theorem 3.1.11 *Let $u^{(0)} = \underline{u}$ on $\bar{\Omega}$ with $u^{(0)} = h$ on $\partial\Omega$. Consider the iteration (3.6). Then $u^{(n)}$ converges to u in $C^2(\bar{\Omega})$, where u is a solution of equation (3.1) in $\mathcal{A}$. If v is any other solution of (3.1) in $\mathcal{A}$ then $u \leq v$ on $\bar{\Omega}$. If $u^{(0)} = \bar{u}$ on $\bar{\Omega}$ with $\bar{u} = h$ on $\partial\Omega$, then the same conclusion holds except that $u \geq v$ on $\bar{\Omega}$.*

Proof: We need to prove that the sequence is **monotonic**, **bounded** and **converging** to a solution. We will prove monotonicity and boundness by induction.

From (3.6) for $n = 0$ we can write

$$-\triangle u^{(1)} + cu^{(1)} = f(u^{(0)}) + cu^{(0)}, \tag{3.7}$$

and $u^{(1)} = u^{(0)}$ on $\partial\Omega$.

Because $u^{(0)} = \underline{u}$ and $\underline{u}$ is a subsolution of (3.1) then

$$-\triangle u^{(0)} \leq f(u^{(0)}).$$

Adding $cu^{(0)}$ on both sides of the above inequality, we get

$$-\triangle u^{(0)} + cu^{(0)} \leq f(u^{(0)}) + cu^{(0)}. \tag{3.8}$$

Using this inequality in (3.7) and rearranging

$$-\triangle u^{(0)} + cu^{(0)} + \triangle u^{(1)} - cu^{(1)} \leq 0 \text{ on } \Omega.$$

Rearranging and multiplying the above result by -1, we have

$$-\triangle(u^{(1)} - u^{(0)}) + c(u^{(1)} - u^{(0)}) \geq 0 \text{ on } \Omega.$$

Since $u^{(1)} = u^{(0)}$ on $\partial\Omega$ that is $u^{(1)} - u^{(0)} = 0$ on $\partial\Omega$, we conclude that $u^{(1)} - u^{(0)} \geq 0$ or $u^{(1)} \geq u^{(0)}$ on Ω by Lemma3.1.8. As $u^{(1)} \geq u^{(0)}$ on Ω we can write from **[A1]** that

$$-c(u^{(1)} - u^{(0)}) \leq f(u^1) - f(u^0),$$

that is

$$f(u^1) \geq f(u^0) - c(u^{(1)} - u^{(0)}). \tag{3.9}$$

Now from (3.7) we have

$$-\triangle u^{(1)} + cu^{(1)} = f(u^{(0)}) + cu^{(0)}.$$

Adding $-cu^{(1)}$ on both sides, we obtain

$$-\triangle u^{(1)} = f(u^{(0)}) + cu^{(0)} - cu^{(1)}.$$

From (3.9) we see

$$-\triangle u^{(1)} \leq f(u^{(1)}).$$

Adding $-f(u^{(1)})$ on both sides, we get

$$-\triangle u^{(1)} - f(u^{(1)}) \leq 0,$$

and we conclude $u^{(1)}$ is a subsolution of (3.1).

Suppose that for some n, $u^{(n)} \geq u^{(n-1)}$ and $u^{(n)}$ is a subsolution of (3.1). We will prove the same is true for $n+1$. From (3.6) we can write for n

$$-\triangle u^{(n+1)} + cu^{(n+1)} = f(u^{(n)}) + cu^{(n)}. \tag{3.10}$$

Now $u^{(n)}$ is a subsolution so

$$-\triangle u^{(n)} - f(u^{(n)}) \leq 0.$$

Adding $cu^{(n)} + f(u^{(n)})$ on both sides, we get

$$-\triangle u^{(n)} + cu^{(n)} \leq f(u^{(n)}) + cu^{(n)},$$

and with the help of (3.10) we can write

$$-\triangle u^{(n)} + cu^{(n)} \leq -\triangle u^{(n+1)} + cu^{(n+1)}.$$

Rearranging and multiplying by -1, we get

$$-(\triangle u^{(n+1)} - \triangle u^{(n)}) + c(u^{(n+1)} - u^{(n)}) \geq 0.$$

By the construction of iteration $u^{(n+1)} = u^{(n)}$ on $\partial\Omega$ that is $u^{(n+1)} - u^{(n)} = 0$ on $\partial\Omega$ hence we conclude by Lemma3.1.8that $\quad u^{(n+1)} \geq u^{(n)}$ on Ω.

Now we will show that $u^{(n+1)}$ is a subsolution. As $u^{(n+1)} \geq u^{(n)}$ on Ω we can write from **[A1]** that

$$-c(u^{(n+1)} - u^{(n)}) \leq f(u^{n+1}) - f(u^{n}),$$

that is

$$f(u^{n+1}) \geq f(u^{n}) - c(u^{(n+1)} - u^{(n)}). \tag{3.11}$$

Now from (3.6) we have

$$-\triangle u^{(n+1)} + cu^{(n+1)} = f(u^{(n)}) + cu^{(n)}.$$

Adding $-cu^{(n+1)}$ on both sides we get

$$-\triangle u^{(n+1)} = f(u^{(n)}) + cu^{(n)} - cu^{(n+1)}.$$

By (3.11) we can write

$$-\triangle u^{(n+1)} \leq f(u^{(n+1)}).$$

Adding $-f(u^{(n+1)})$ on both sides we get

$$-\triangle u^{(n+1)} - f(u^{(n+1)}) \leq 0.$$

Hence $u^{(n+1)}$ is a subsolution of (3.1).

We have proved that $u^{(n)}$ is monotonically increasing from subsolution $\underline{u}$ and the sequence $\{u^{(n)}\}$ is bounded below by $\underline{u}$ in Ω. Now we need to show that for all $n \in \mathcal{N}$, $u^{(n)} \leq \bar{u}$.

From (3.7) we know

$$-\triangle u^{(1)} + cu^{(1)} = f(u^{(0)}) + cu^{(0)}. \tag{3.12}$$

Also we know $\bar{u}$ is a supersolution, so

$$-\triangle \bar{u} - f(\bar{u}) \geq 0.$$

Adding $f(\bar{u}) + c\bar{u}$ on both sides of the above inequality, we get

$$-\triangle \bar{u} + c\bar{u} \geq f(\bar{u}) + c\bar{u}. \tag{3.13}$$

Now subtracting (3.12) from (3.13) we have

$$-\triangle(\bar{u}-u^{(1)})+c(\bar{u}-u^{(1)}) \geq f(\bar{u})-f(u^{(0)})+c(\bar{u}-u^{(0)}). \tag{3.14}$$

By property **[A1]** we know $-c(\bar{u}-u^{(0)}) \leq f(\bar{u})-f(u^{(0)})$, so (3.14) implies

$$-\triangle(\bar{u}-u^{(1)})+c(\bar{u}-u^{(1)}) \geq 0.$$

As $\bar{u} \geq u^{(1)} = h$ on $\partial\Omega$ so by Lemma3.1.8we conclude that $\bar{}u \geq u^{(1)}$ on Ω.

Assume that for some n, $\bar{u} \geq u^{(n)}$ we will prove it is true for $n+1$, that is $\bar{u} \geq u^{(n+1)}$. Now using the definition of equation for $u^{(n+1)}$ and subtracting it from (3.13) we get

$$-\triangle(\bar{u}-u^{(n+1)})+c(\bar{u}-u^{(n+1)}) \geq f(\bar{u})-f(u^{(n)})+c(\bar{u}-u^{(n)}). \tag{3.15}$$

By property **[A1]** we know $-c(\bar{u}-u^{(n)}) \leq f(\bar{u})-f(u^{(n)})$, so (3.15) implies

$$-\triangle(\bar{u}-u^{(n+1)})+c(\bar{u}-u^{(n+1)}) \geq 0.$$

As $\bar{u} \geq u^{(n+1)} = h$ on $\partial\Omega$, Lemma3.1.8ensures that $\bar{}u \geq u^{(n+1)}$ on Ω. Hence we conclude that

$$\underline{u} \leq u^{(n)} \leq \bar{u} \text{ on } \bar{\Omega}, n \geq 0. \tag{3.16}$$

Since the sequence is bounded above and monotonic, the point wise limits exist in $\bar{\Omega}$, that is

$$\lim_{n\to\infty} u^{(n)}(\xi) = u(\xi).$$

Now we want to show that $\{u^{(n)}\}$ converges to a solution of the PDE (3.1) in $\mathcal{C}^2(\bar{\Omega})$. To start we show that $\{u^{(n)}\}$ is uniformly bounded in $\mathcal{C}^{2+\mu}(\bar{\Omega})$. From the point wise convergence of $\{u^{(n)}\}$ and the continuity property of f, the sequence $\{f(u^{(n)})\}$ converges point wise to $f(u)$ as $n \to \infty$. Also we know $\underline{u}(\xi) \leq u^{(n)}(\xi) \leq \bar{u}(\xi)$ for all $n \in \mathcal{N}$ and for all $\xi \in \bar{\Omega}$, where $\bar{\Omega}$ is bounded domain. So, there exist a maximum of $\bar{u}(\xi)$ say M_1 and a minimum of $\underline{u}(\xi)$ say M_2 on $\bar{\Omega}$ such that $M_2 \leq u^{(n)}(\xi) \leq M_1$ for all $n \in \mathcal{N}$. As f is Hölder continuous $\bar{\Omega}$ so f is continuous on $\bar{\Omega}$. Hence there exists a constant C_1 such that

$$|| \ f(u^{(n)}(\xi)) \ ||_{\mathcal{L}_\infty} \leq C_1 \text{ for all } n \in \mathcal{N} \text{ and for all } \xi \in \bar{\Omega}.$$

Now

$$|| \ f(u^{(n)}(\xi)) \ ||_{\mathcal{L}_p} = \Big(\int\limits_{\Omega} | \ f(u^{(n)}(\xi)) \ |^p \ d\xi \Big)^{\frac{1}{p}} \leq \Big(\ || \ f(u^{(n)}(\xi)) \ ||^p_{\mathcal{L}_\infty} | \ \Omega \ | \ \Big)^{\frac{1}{p}},$$

that is $|| \ f(u^{(n)}(\xi)) \ ||_{\mathcal{L}_p} \leq C_2$, where $C_2 = \Big(\ || \ f(u^{(n)}(\xi)) \ ||^p_{\mathcal{L}_\infty} | \ \Omega \ | \ \Big)^{\frac{1}{p}}$ which is finite. Hence $\{f(u^{(n)})\}$ is uniformly bounded in $\mathcal{L}_p(\Omega)$ for every $p \geq 1$. Hence by Lemma 3.1.4we know $\{u^{(n)}\}$ is uniformly bounded in $\mathcal{W}^2_p(\Omega)$. We choose $p > N$ so that $\mu = 1 - \frac{N}{p}$, then by Lemma3.1.5, $\{u^{(n)}\}$ is uniformly bounded in $\mathcal{C}^{1+\mu}(\bar{\Omega})$. Furthermore by Lemma3.1.6, $\{f(u^{(n)})\}$ is uniformly bounded in $\mathcal{C}^{\mu}(\bar{\Omega})$. So we conclude by Lemma 3.1.7that $\{u^{(n)}\}$ is uniformly bounded in $\mathcal{C}^{2+\mu}(\bar{\Omega})$. Hence by Theorem3.1.9there exists a subsequence of $\{u^{(n)}\}$ which converges and Lemma3.1.10and Remark1 confirm that this subsequence converge in $\mathcal{C}^2(\bar{\Omega})$ to u^*. But since $\{u^{(n)}\}$ converges point wise to u and this sequence is monotonic, we have $u = u^*$ and for the same reason the whole sequence $\{u^{(n)}\}$ converges in $\mathcal{C}^2(\bar{\Omega})$ to u. Now $f(u^{(n)}) \to f(u)$ as $n \to \infty$ which implies that u is a solution of (3.1). ■

3.2 Linearized single domain method to solve $u^{''} = f(\xi, u, u^{'})$

Consider the Dirichlet problem

$$-u^{''} + f(\xi, u, u^{'}) = 0, \quad u(a) = u(b) = 0. \tag{3.17}$$

Cherpion developed an iterative approach in [6] and [5] to solve the BVP that has the form (3.17). Here we fully explain the proof of this theorem filling in all the details of the proof. In order to prove the theorem on a single domain first we have to prove some lemmas.

The subsolution and supersolution of (3.17) are defined in Definition3.2.1and Definition3.2.2respectively.

Definition 3.2.1 (Subsolution:) *A function $\underline{\alpha} \in \mathcal{C}^2([a, b])$ is a subsolution of* (3.17) *if*

(I) for all $\xi \in [a, b]$, $\underline{\alpha}^{''}(\xi) \geq f(\xi,\underline{\alpha}(\xi),\underline{\alpha}^{'}(\xi))$;

(II) $\underline{\alpha}(a) \leq 0$, $\underline{\alpha}(b) \leq 0$.

Definition 3.2.2 (Supersolution:) *A function $\bar{\beta} \in \mathcal{C}^2([a, b])$ is supersolution of (3.17) if*

(I) for all $\xi \in [a, b]$, $\bar{\beta}^{''}(\xi) \leq f(\xi, \bar{\beta}(\xi), \bar{\beta}^{'}(\xi))$;

(II) $\bar{\beta}(a) \geq 0$, $\bar{\beta}(b) \geq 0$.

Let us define some properties which we will be using in the theorem.

Definition 3.2.3 (Properties:)[C1] *Let $\underline{\alpha}$ and $\bar{\beta} \in \mathcal{C}^2([a, b])$ be the subsolution and supersolution of* (3.17) *such that $\underline{\alpha} \leq \bar{\beta}$ and define the set $\mathcal{D}$ as*

$$\mathcal{D} = \{(\xi, u, v) \in [a, b] \times \mathcal{R}^2 | \underline{\alpha}(\xi) \leq u \leq \bar{\beta}(\xi)\}.$$

[C2] ***One Sided Lipschitz Condition in*** u***:*** *Suppose $f : \mathcal{D} \to \mathcal{R}$ is a continuous function. We assume there exist $M \geq 0$ such that for all (ξ, u_1, v), $(\xi, u_2, v) \in D$ and for all $u_1 \leq u_2$,*

$$f(\xi, u_2, v) - f(\xi, u_1, v) \leq M(u_2 - u_1).$$

[C3] ***Lipschitz Condition in*** $u^{'}$***:*** *Suppose $f : \mathcal{D} \to \mathcal{R}$ is a continuous function. We assume there exist $N \geq 0$ such that for all (ξ, u, v_1), $(\xi, u, v_2) \in D$,*

$$\mid f(\xi, u, v_2) - f(\xi, u, v_1) \mid \leq N \mid v_2 - v_1 \mid . \tag{3.18}$$

Under these properties for given subsolution $\underline{\alpha}$ and supersolution $\bar{\beta}$ the iteration schemes to solve (3.17) are defined for $n = 0, 1, 2, ...$ as

$$- \alpha^{''}_{n+1} + \sqrt[3]{l}K(\xi)\alpha^{'}_{n+1} + l\alpha_{n+1} = -f(\xi, \alpha_n, \alpha^{'}_n) + \sqrt[3]{l}K(\xi)\alpha^{'}_n + l\alpha_n, \tag{3.19}$$

$$\alpha_0 = \alpha,$$

and

$$- \beta^{''}_{n+1} + \sqrt[3]{l}K(\xi)\beta^{'}_{n+1} + l\beta_{n+1} = -f(\xi, \beta_n, \beta^{'}_n) + \sqrt[3]{l}K(\xi)\beta^{'}_n + l\beta_n, \tag{3.20}$$

$$\beta_0 = \beta,$$

where $l \geq 0$ is a constant (to be specifies later) and $K(\xi) \in \mathcal{C}([a, b])$ is antisymmetric function.

A maximum principle for a second order linear elliptic differential equation is provided in the following lemma.

Lemma 3.2.4 *Assume $w \in \mathcal{C}^2$, $l \in \mathcal{R}^+$ and $K(\xi)$ is a continuous function on Ω. Define the elliptic differential operator $\mathcal{L}$ in a bounded domain Ω with boundary $\partial\Omega$*

as

$$\mathcal{L}w \equiv w^{''} - \sqrt[3]{l}K(\xi)w^{'} - lw.$$

If $\mathcal{L}w \leq 0$ with $w \geq 0$ on $\partial\Omega$ then $w \geq 0$ on $\bar{\Omega}$.

Proof: Assume $\mathcal{L}w$ is negative. By contradiction we assume that w is not positive in Ω. Since w is continuous, differentiable and non-negative on the boundary there must be a minimum negative value of w. Assume this occurs at ξ_0 in the interior of Ω. So at this point $w^{'}(\xi_0) = 0$ and $w^{''}(\xi_0) \geq 0$.

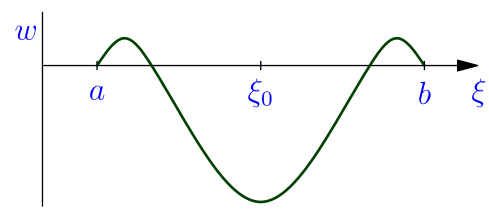


Figure 3.1: Contradictory shape of function w.

Evaluating the differential equation at ξ_0 we have $-lw(\xi_0) \leq 0$ which is a contradiction since $w \leq 0$. Hence we conclude that $w \geq 0$ in Ω. ■

Recalling the following basic result for initial value problems from [18]

Lemma 3.2.5 *Consider the initial value problem*

$$u^{''} + p(\xi)u^{'} + q(\xi)u = f(\xi), \tag{3.21}$$
$$u(\xi_0) = u_0, \qquad u^{'}(\xi_0) = u_0^{'}.$$

If the functions $p(\xi)$, $q(\xi)$ and $f(\xi)$ are continuous on $[a, b]$ and ξ_0 is any point of the interval $[a, b]$, then there exists a unique solution u of the problem (3.21) and that solution exists throughout the interval $[a, b]$.

Proof: The proof of this lemma can be found in [18] or [2]. ■

This helps to prove the followings.

Lemma 3.2.6 *If* $z_1(\xi)$ *and* $z_2(\xi)$ *for all* $\xi \in [a, b]$ *satisfy*

$$z_1''(\xi) - \sqrt[3]{l}K(\xi)z_1'(\xi) - lz_1(\xi) = 0, \qquad z_1(a) = 0, z_1'(a) = 1, \tag{3.22}$$

$$z_2''(\xi) - \sqrt[3]{l}K(\xi)z_2'(\xi) - lz_2(\xi) = 0, \qquad z_2(b) = 0, z_2'(b) = -1, \tag{3.23}$$

respectively, where $K(\xi) = -K(a + b - \xi)$*, then* $z_2(\xi) = z_1(a + b - \xi)$*.*

Proof: We are given that $z_1(\xi)$ is a solution of equation (3.22) for all $\xi \in [a, b]$. As $\xi \in [a, b]$ then $a < \xi$ or $-a > -\xi$. Adding $a + b$ on both sides of the inequality we get $b > a + b - \xi$. Also, $b > \xi$ or $-b < -\xi$. Adding $a + b$ on both sides of the inequality we get $a < a + b - \xi$. So $(a + b - \xi) \in [a, b]$, for all $\xi \in [a, b]$. Now evaluating equation (3.22) at $\bar{\xi} = (a + b - \xi) \in [a, b]$, we have

$$\frac{d^2 z_1(\bar{\xi})}{d(\bar{\xi})^2} - \sqrt[3]{l}K(\bar{\xi})\frac{dz_1(\bar{\xi})}{d(\bar{\xi})} - lz_1(\bar{\xi}) = 0.$$

As $\bar{\xi} = (a + b - \xi)$, we can write

$$\frac{d}{d(a+b-\xi)}\left\{\frac{dz_1(a+b-\xi)}{d(a+b-\xi)}\right\} - \sqrt[3]{l}K(a+b-\xi)\frac{dz_1(a+b-\xi)}{d(a+b-\xi)} - lz_1(a+b-\xi) = 0.$$

Now differentiating using the chain rule, we get

$$-\frac{d}{d\xi}\left\{-\frac{dz_1(a+b-\xi)}{d\xi}\right\} + \sqrt[3]{l}K(a+b-\xi)\frac{dz_1(a+b-\xi)}{d\xi} - lz_1(a+b-\xi) = 0.$$

Using $K(\xi) = -K(a + b - \xi)$, we have

$$\frac{d^2 z_1(a+b-\xi)}{d\xi^2} - \sqrt[3]{l}K(\xi)\frac{dz_1(a+b-\xi)}{d(\xi)} - lz_1(a+b-\xi) = 0.$$

The initial condition is $z_1(\bar{\xi})|_{\bar{\xi}=a} = 0$ or $z_1(a + b - \xi)|_{a+b-\xi=a} = 0$ which implies $z_1(a + b - \xi)|_{\xi=b} = 0$. Likewise $\frac{dz_1(\bar{\xi})}{d\bar{\xi}}|_{\bar{\xi}=a} = 1$ or $\frac{dz_1(a+b-\xi)}{d(a+b-\xi)}|_{a+b-\xi=a} = 1$ and using the chain rule to evaluate the derivative at ξ we get $-\frac{dz_1(a+b-\xi)}{d\xi}|_{\xi=b} = 1$ which implies $\frac{dz_1(a+b-\xi)}{d\xi}|_{\xi=b} = -1$. So $z_1(a+b-\xi)$ is a solution of (3.23), hence by the uniqueness Lemma3.2.5we conclude $\quad z_2(\xi) = z_1(a+b-\xi)$. ■

In the following lemma we prove the solutions of equations (3.22) and (3.23) are always positive.

Lemma 3.2.7 *Assume $K(\xi) \in \mathcal{C}([a,b])$ is such that $K(a) > 0$ and for all $\xi \in [a,b]$, $K(\xi) = -K(a+b-\xi)$. Furthermore, assume z_1 is the solution of equation*

$$z_1^{''} - \sqrt[3]{l}K(\xi)z_1^{'} - lz_1 = 0, \qquad z_1(a) = 0, z_1^{'}(a) = 1. \tag{3.24}$$

Then for $\xi \in (a,b]$, $z_1(\xi) > 0$ and $z_1^{'}(\xi) > 0$. Furthermore we have $z_2 > 0$ and $z_2^{'}(\xi) < 0$, where z_2 is a solution of equation

$$z_2^{''} - \sqrt[3]{l}K(\xi)z_2^{'} - lz_2 = 0, \qquad z_2(b) = 0, z_2^{'}(b) = -1. \tag{3.25}$$

Proof: We proceed by contradiction. Assume there exists a point $\xi_0 \in (a,b]$ such that $z_1^{'}(\xi_0) = 0$. As the function z_1 is increasing initially, to have $z_1^{'}(\xi_0) = 0$ we would need $z_1^{''}(\xi_0) \leq 0$, see Figure3.2. But at ξ_0 we have $z_1^{''}(\xi_0) - \sqrt[3]{l}K(\xi_0)z_1^{'}(\xi_0) - lz_1(\xi_0) = 0$.

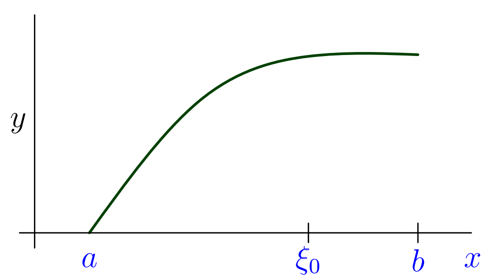


Figure 3.2: Sample shape of function z_1.

This implies $z_1^{''}(\xi_0) = lz_1(\xi_0) > 0$, which is a contradiction. Hence the result is proved. As $z_2(\xi) = z_1(a+b-\xi)$, we also have that for all $\xi \in [a,b)$, $z_2(\xi) > 0$ and $z_2^{'}(\xi) < 0$, because $\frac{dz_2(\xi)}{d\xi} = -z_1^{'}(a+b-\xi)$. ■

Lemma 3.2.8 *If for all $\xi \in [a,b]$, $z_1(\xi)$ and $z_2(\xi)$ are the solution of equations (3.22) and (3.23) respectively, with $K(\xi) = -K(a+b-\xi)$ then z_1 and z_2 are linearly independent.*

Proof: We assume by contradiction that z_1 and z_2 are not linearly independent. That is there exists a $C \in \mathcal{R}-\{0\}$ such that $z_1(\xi) = Cz_2(\xi)$ or $z_1(\xi) = Cz_1(a+b-\xi)$, since we know $z_2(\xi) = z_1(a+b-\xi)$. At $\xi = a$ we know $z_1(a) = 0$ thus $0 = Cz_1(a+b-a)$ and hence $z_1(b) = 0$ as $C \neq 0$. This is a contradiction, since we know from Lemma 3.2.7that for all $\xi \in (a,b]$, $z_1(\xi) > 0$. Hence $z_1(\xi)$ and $z_2(\xi)$ are linearly independent.
■

Lemma 3.2.9 *Consider the problem,*

$$w'' - \sqrt[3]{l}K(\xi)w' - lw = h(\xi), \qquad w(a) = A, w(b) = B, \tag{3.26}$$

Where $A, B \in \mathcal{R}$. If z_1 and z_2 are the solution of the equations (3.22) and (3.23) respectively, then the solution $w(\xi)$ of the equation (3.26) is

$$\begin{aligned} w(\xi) =& z_2(\xi)\Big[\int_a^\xi \frac{z_1(s)h(s)}{z_2'(s)z_1(s) - z_2(s)z_1'(s)}ds\Big] + z_1(\xi)\Big[\int_\xi^b \frac{z_2(s)h(s)}{z_2'(s)z_1(s) - z_2(s)z_1'(s)}ds\Big] \\ &+ \frac{Az_2(\xi)}{z_2(a)} + \frac{Bz_1(\xi)}{z_1(b)}. \end{aligned} \tag{3.27}$$

Proof: To prove that (3.27) is the solution of equation (3.26), we will substitute w and its derivatives into equation (3.26). Since

$$\begin{aligned} w(\xi) = & z_2(\xi)\Big[\int_a^\xi \frac{z_1(s)h(s)}{z_2'(s)z_1(s) - z_2(s)z_1'(s)}ds\Big] + z_1(\xi)\Big[\int_\xi^b \frac{z_2(s)h(s)}{z_2'(s)z_1(s) - z_2(s)z_1'(s)}ds\Big] \\ &+ \frac{Az_2(\xi)}{z_2(a)} + \frac{Bz_1(\xi)}{z_1(b)}, \end{aligned}$$

then differentiating with respect to ξ we have

$$w^{'}(\xi) = z_2^{'}(\xi)\Big[\int\limits_a^\xi \frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] + \frac{z_1(\xi)z_2(\xi)h(\xi)}{z_2^{'}(\xi)z_1(\xi) - z_2(\xi)z_1^{'}(\xi)}$$
$$+ z_1^{'}(\xi)\Big[\int\limits_\xi^b \frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] - \frac{z_1(\xi)z_2(\xi)h(\xi)}{z_2^{'}(\xi)z_1(\xi) - z_2(\xi)z_1^{'}(\xi)}$$
$$+ \frac{Az_2^{'}(\xi)}{z_2(a)} + \frac{Bz_1^{'}(\xi)}{z_1(b)}.$$

Canceling like terms we get

$$w^{'}(\xi) = z_2^{'}(\xi)\Big[\int\limits_a^\xi \frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] + z_1^{'}(\xi)\Big[\int\limits_\xi^b \frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big]$$
$$+ \frac{Az_2^{'}(\xi)}{z_2(a)} + \frac{Bz_1^{'}(\xi)}{z_1(b)}.$$

Again differentiating with respect to ξ we have

$$w^{''}(\xi) = z_2^{''}(\xi)\Big[\int\limits_a^\xi \frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] + \frac{z_1(\xi)z_2^{'}(\xi)h(\xi)}{z_2^{'}(\xi)z_1(\xi) - z_2(\xi)z_1^{'}(\xi)}$$
$$+ z_1^{''}(\xi)\Big[\int\limits_\xi^b \frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] - \frac{z_1^{'}(\xi)z_2(\xi)h(\xi)}{z_2^{'}(\xi)z_1(\xi) - z_2(\xi)z_1^{'}(\xi)}$$
$$+ \frac{Az_2^{''}(\xi)}{z_2(a)} + \frac{Bz_1^{''}(\xi)}{z_1(b)}.$$

Combining terms we will end up with the formula

$$w^{''}(\xi) = z_2^{''}(\xi)\Big[\int\limits_a^\xi \frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] + z_1^{''}(\xi)\Big[\int\limits_\xi^b \frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big]$$
$$+ h(\xi) + \frac{Az_2^{''}(\xi)}{z_2(a)} + \frac{Bz_1^{''}(\xi)}{z_1(b)}.$$

Substituting these expressions into (3.26) we have

$$\begin{aligned}
&w^{''}-\sqrt[3]{l}K(\xi)w^{'}-lw\\
&=\Bigg[z_2^{''}(\xi)\Big[\int\limits_a^{\xi}\frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]+z_1^{''}(\xi)\Big[\int\limits_{\xi}^{b}\frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]\\
&+h(\xi)+\frac{Az_2^{''}(\xi)}{z_2(a)}+\frac{Bz_1^{''}(\xi)}{z_1(b)}\Bigg]-\sqrt[3]{l}K(\xi)\Bigg[z_2^{'}(\xi)\Big[\int\limits_a^{\xi}\frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]\\
&+z_1^{'}(\xi)\Big[\int\limits_{\xi}^{b}\frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]+\frac{Az_2^{'}(\xi)}{z_2(a)}+\frac{Bz_1^{'}(\xi)}{z_1(b)}\Bigg]\\
&-l\Bigg[z_2(\xi)\Big[\int\limits_a^{\xi}\frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]+z_1(\xi)\Big[\int\limits_{\xi}^{b}\frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]\\
&+\frac{Az_2(\xi)}{z_2(a)}+\frac{Bz_1(\xi)}{z_1(b)}\Bigg],\\
&=\Bigg[z_2^{''}-\sqrt[3]{l}K(\xi)z_2^{'}-lz_2\Bigg]\Big[\int\limits_a^{\xi}\frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]\\
&+\Bigg[z_1^{''}-\sqrt[3]{l}K(\xi)z_1^{'}-lz_1\Bigg]\Big[\int\limits_a^{\xi}\frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]+h(\xi)\\
&+\frac{A}{z_2(a)}\Bigg[z_2^{''}-\sqrt[3]{l}K(\xi)z_2^{'}-lz_2\Bigg]+\frac{B}{z_1(b)}\Bigg[z_1^{''}-\sqrt[3]{l}K(\xi)z_1^{'}-lz_1\Bigg],\\
&=h(\xi).
\end{aligned}$$

The last equality follows from the assumption on z_1 and z_2 from (3.22) and (3.23). Hence we conclude that (3.27) is the solution of equation (3.26). ■

Lemma3.2.10provides a technical result which is needed to prove Lemma 3.2.11.

Lemma 3.2.10 *Let $M, N, l \in \mathcal{R}^+$ and let $\mid K\mid_\infty$ be the maximum value of $K(\xi)$ on*

$[a, b]$ *and* ξ_2 *a constant which satisfies* $\xi_2 > a$*. Then for* l *large enough,*

$$C \equiv (M - l) + (N + \sqrt[3]{l}|K|_\infty)\sqrt[3]{l}|K|_\infty + \frac{(N + \sqrt[3]{l}|K|_\infty)^2 l}{l - M} < \frac{-2\sqrt[3]{l}|K|_\infty}{\xi_2 - a}. \quad (3.28)$$

Proof: We rewrite the inequality (3.28) as

$$M + (N + \sqrt[3]{l}|K|_\infty)\sqrt[3]{l}|K|_\infty + \frac{(N + \sqrt[3]{l}|K|_\infty)^2 l}{l - M} + \frac{2\sqrt[3]{l}|K|_\infty}{\xi_2 - a} < l.$$

If l is positive then this is equivalent to

$$\frac{M}{l} + \frac{N\sqrt[3]{l} \mid K \mid_\infty}{l} + \frac{\sqrt[3]{l^2} \mid K \mid_\infty^2}{l} + \frac{(N + \sqrt[3]{l}|K|_\infty)^2}{l - M} + \frac{2\sqrt[3]{l}|K|_\infty}{(\xi_2 - a)l} < 1. \quad (3.29)$$

Now for l large enough left hand side of the inequality (3.29) decreases to zero which is less than 1. Hence for l large enough inequality (3.28) is true. ■

Lemma 3.2.11 *Let* $M, N \in \mathcal{R}^+$ *and* z_1 *be the solution of equation (3.24). Assume* $K \in \mathcal{C}([a, b])$ *is such that* $K(a) > 0$ *and for all* $\xi \in [a, b]$, $K(\xi) = -K(a + b - \xi)$*. Also let* $l > \max(M, \frac{N^3}{(K(a))^3})$ *then, we have for all* $\xi \in [a, b]$,

$$(M - l)z_1(\xi) + (N - \sqrt[3]{l}K(\xi))z_1'(\xi) \le 0. \quad (3.30)$$

Proof: Let

$$P(\xi) = (M - l)z_1(\xi) + (N - \sqrt[3]{l}K(\xi))z_1'(\xi).$$

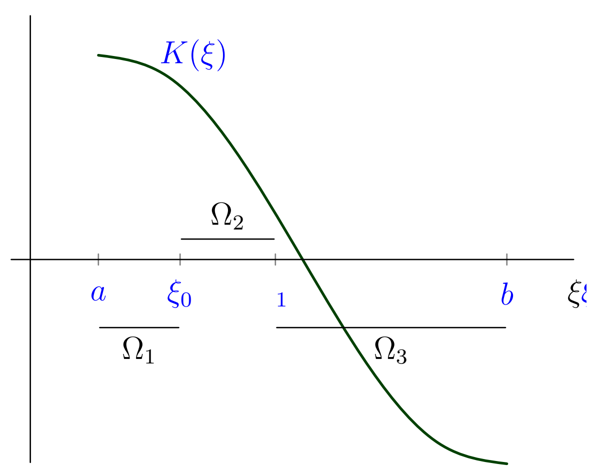


Figure 3.3: Domain partition for $P(\xi)$

For our convenience we split the domain $\Omega = [a, b]$ into three different subdomains which are $\Omega_1 = [a, \xi_0]$, $\Omega_2 = (\xi_0, \xi_1]$ and $\Omega_3 = (\xi_1, b]$ as shown in Figure3.3. At $\xi = a$,

$$P(a) = (M - l)z_1(a) + (N - \sqrt[3]{l}K(a))z_1'(a) = N - \sqrt[3]{l}K(a) < 0$$

because $l > \max(M, \frac{N^3}{(K(a))^3})$. As $K(a) > 0$ and $K(\xi)$ is continuous on $[a, b]$ then there exists a $\xi_1 \in (a, b)$ such that for all $\xi \in [a, \xi_1]$, $K(\xi) > 0$. As $P(a) < 0$ and $P(\xi)$ is continuous on $[a, b]$ we can find a $\bar{\xi} \in [a, b]$ with either $\bar{\xi} < \xi_1$ or $\bar{\xi} \geq \xi_1$ such that $P(\xi) < 0$ on $[a, \bar{\xi}]$. If $\bar{\xi} < \xi_1$, we let $\xi_0 = \bar{\xi}$ and so $P(\xi) < 0$ for all $\xi \in [a, \xi_0]$. If $\bar{\xi} \geq \xi_1$, we choose ξ_0 such that $\xi_0 < \xi_1 \leq \bar{\xi}$ and so $P(\xi) < 0$ for all $\xi \in \Omega_1 = [a, \xi_0]$. Now we have to prove that for all $\xi \in \Omega_2 = (\xi_0, \xi_1]$, $P(\xi) < 0$. To do this we define

$$g(\xi) := (M - l)z_1(\xi) + (N + \sqrt[3]{l}|K|_\infty)z_1'(\xi).$$

Since $z_1'(\xi) > 0$ and $|K|_\infty \geq \pm K(\xi)$, we have $P(\xi) \leq g(\xi)$ on $[a, b]$ and hence $P(\xi) \leq g(\xi)$ on $(\xi_0, \xi_1]$. We will prove by contradiction that $g(\xi) < 0$ on $(\xi_0, \xi_1]$ and hence $P(\xi) < 0$ on $(\xi_0, \xi_1]$. If it is not true, then we can find $\xi_2 \in (\xi_0, \xi_1]$ such that $g(\xi_2) \geq 0$. If $g(\xi_2) \geq 0$ then

$$\begin{aligned}
&(M - l)z_1(\xi_2) + (N + \sqrt[3]{l}|K|_\infty)z_1'(\xi_2) \geq 0 \\
\Rightarrow &(M - l)z_1(\xi_2) \geq -(N + \sqrt[3]{l}|K|_\infty)z_1'(\xi_2) \\
\Rightarrow &(l - M)z_1(\xi_2) \leq (N + \sqrt[3]{l}|K|_\infty)z_1'(\xi_2) \\
\Rightarrow &z_1(\xi_2) \leq \frac{(N + \sqrt[3]{l}|K|_\infty)}{(l - M)}z_1'(\xi_2), \qquad \text{since } l > M.
\end{aligned}$$

By equation (3.24) $z_1'' = \sqrt[3]{l}K(\xi_2)z_1' + lz_1$ and hence,

$$\begin{aligned}
g'(\xi_2) &= (M - l)z_1'(\xi_2) + (N + \sqrt[3]{l}|K|_\infty)z_1''(\xi_2) \\
&= (M - l)z_1'(\xi_2) + (N + \sqrt[3]{l}|K|_\infty)(\sqrt[3]{l}K(\xi_2)z_1'(\xi_2) + lz_1(\xi_2)) \\
&\leq (M - l)z_1'(\xi_2) + (N + \sqrt[3]{l}|K|_\infty)\Big(\sqrt[3]{l}K(\xi_2)z_1'(\xi_2) + \frac{(N + \sqrt[3]{l}|K|_\infty)l}{l - M}z_1'(\xi_2)\Big) \\
&\equiv Cz_1'(\xi_2), \qquad (3.31)
\end{aligned}$$

where $C = (M-l) + (N + \sqrt[3]{l}|K|_\infty)\sqrt[3]{l}|K|_\infty + \frac{(N+\sqrt[3]{l}|K|_\infty)^2 l}{l-M}$. From Lemma3.2.10we know that for l large enough $C < \frac{-2\sqrt[3]{l}|K|_\infty}{\xi_2 - a}$. Also we already know that $z_1(\xi) > 0$ and $z_1^{'}(\xi) > 0$ for all $\xi \in (a,b]$ and hence on $(\xi_0, \xi_1]$. Also $z_1^{''} - \sqrt[3]{l}K(\xi)z_1^{'} - lz_1 = 0$, which implies $z_1^{''} = \sqrt[3]{l}K(\xi)z_1^{'} + lz_1 > 0$ and hence $z_1^{'}(\xi)$ is increasing for all $\xi \in (\xi_0, \xi_1]$. Therefore we have $z_1^{'}(\xi_2) > z_1^{'}(a) = 1$. This leads to the contradiction as we will now show. Direct integration gives

$$\int_a^{\xi_2} g^{'} ds = g(\xi_2) - g(a).$$

Adding $g(a)$ on both sides of the above equation and rearranging we have

$$g(\xi_2) = g(a) + \int_a^{\xi_2} g^{'} ds.$$

With the help of (3.31) and our assumption that $g(\xi_2) \geq 0$ we can write

$$\begin{aligned} 0 \leq g(\xi_2) \leq g(a) &+ \int_a^{\xi_2} C z_1^{'}(s) ds \\ &< N + \sqrt[3]{l}|K|_\infty + \int_a^{\xi_2} C z_1^{'}(s) ds \\ &< 2\sqrt[3]{l}|K|_\infty - \frac{2\sqrt[3]{l}|K|_\infty}{(\xi_2 - a)}(\xi_2 - a) = 0, \end{aligned}$$

which is the desired contradiction. Hence for all $\xi \in (\xi_0, \xi_1]$ we have $P(\xi) < 0$.

Now on $(\xi_1, b]$ we also have $P(\xi) < g(\xi)$. So we will prove that $g(\xi) < 0$ on $(\xi_1, b]$ and hence $P(\xi) < 0$. If $g(\xi) \geq 0$ on $(\xi_1, b]$ then from (3.31) we know $g^{'}(\xi) < Cz_1^{'}$ with $C < 0$. We also know that $z_1^{'}(\xi) > 0$ on $(\xi_1, b]$, so $g^{'}(\xi) < 0$ on $(\xi_1, b]$, that is $g(\xi)$ is decreasing on $(\xi_1, b]$ from a negative value of $g(\xi)$ at ξ_1. So $g(\xi) < 0$ on $(\xi_1, b]$ which is a contradiction. Hence we conclude that for all $\xi \in [a, b]$

$$P(\xi) = (M-l)z_1(\xi) + (N - \sqrt[3]{l}K(\xi))z_1^{'}(\xi) < g(\xi) \leq 0,$$

and the required (3.30) is confirmed. ■

Lemma 3.2.12 *Let $M, N, l \in \mathcal{R}^+$ and assume $K(\xi) = -K(a+b-\xi)$ on $[a,b]$. Let $\mid K \mid_\infty$ be the maximum value of $K(\xi)$ on $[a,b]$. Suppose $w^{'}$ is the derivative of the function w stated in the formula (3.27). Then z_1 satisfies the inequality*

$$(M-l)z_1(\xi) + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))z_1^{'}(\xi) \leq 0$$

and z_2 satisfies the inequality

$$(M-l)z_2(\xi) + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))z_2^{'}(\xi) \leq 0.$$

Proof: As $N \in \mathcal{R}^+$ so $-N < N$, adding $-\sqrt[3]{l}K(\xi)$ we have

$$-N - \sqrt[3]{l}K(\xi) < N - \sqrt[3]{l}K(\xi).$$

We know $z_1^{'}(\xi) \geq 0$ on $[a,b]$, so multiplying both sides of the above inequality by $z_1^{'}(\xi)$,

$$(-N - \sqrt[3]{l}K(\xi))z_1^{'}(\xi) < (N - \sqrt[3]{l}K(\xi))z_1^{'}(\xi).$$

Now adding $(M-l)z_1(\xi)$ on both sides, we get

$$(M-l)z_1(\xi) + (-N - \sqrt[3]{l}K(\xi))z_1^{'}(\xi) < (M-l)z_1(\xi) + (N - \sqrt[3]{l}K(\xi))z_1^{'}(\xi).$$

From Lemma3.2.11($M-l)z_1 + (N - \sqrt[3]{l}K(\xi))z_1^{'} \leq 0$ and $\text{sign}(w^{'}) = \pm 1$ then we have

$$(M-l)z_1(\xi) + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))z_1^{'}(\xi) \leq 0,$$

for any $\xi \in [a,b]$.

To prove the second inequality from Lemma3.2.6we know $z_2(\xi) = z_1(a+b-\xi)$ so $z_2^{'}(\xi) = -z_1^{'}(a+b-\xi)$. So

$$\begin{aligned}
&(M-l)z_2(\xi) + (N - \sqrt[3]{l}K(\xi))z_2^{'}(\xi),\\
=&(M-l)z_1(a+b-\xi) + (N - \sqrt[3]{l}K(\xi))(-z_1^{'}(a+b-\xi)),\\
=&(M-l)z_1(a+b-\xi) + (-N + \sqrt[3]{l}K(\xi))z_1^{'}(a+b-\xi),\\
=&(M-l)z_1(a+b-\xi) + (-N - \sqrt[3]{l}K(a+b-\xi))z_1^{'}(a+b-\xi),\\
<&(M-l)z_1(a+b-\xi) + (N - \sqrt[3]{l}K(a+b-\xi))z_1^{'}(a+b-\xi) \leq 0.
\end{aligned}$$

The last inequality follows from Lemma3.2.12. Similarly

$$\begin{aligned}
&(M-l)z_2(\xi) + (-N - \sqrt[3]{l}K(\xi))z_2^{'}(\xi),\\
=&(M-l)z_1(a+b-\xi) + (-N - \sqrt[3]{l}K(\xi))(-z_1^{'}(a+b-\xi)),\\
=&(M-l)z_1(a+b-\xi) + (N + \sqrt[3]{l}K(\xi))z_1^{'}(a+b-\xi),\\
=&(M-l)z_1(a+b-\xi) + (N - \sqrt[3]{l}K(a+b-\xi))z_1^{'}(a+b-\xi) \leq 0.
\end{aligned}$$

Since $\text{sign}(w^{'}) = \pm 1$, we can write

$$(M-l)z_2(\xi) + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))z_2^{'}(\xi) \leq 0.$$

Hence the lemma is proved. ■

Lemma 3.2.13 *If $h(\xi) \leq 0$, $A, B \in \mathcal{R}_0^+$ and*

$$w(\xi) = z_2(\xi)\Big[\int_a^{\xi} \frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] + z_1(\xi)\Big[\int_{\xi}^{b} \frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] + \frac{Az_2(\xi)}{z_2(a)} + \frac{Bz_1(\xi)}{z_1(b)},$$

where z_1 and z_2 are the solution of equation (3.22) and (3.23) respectively, then

$$(M-l)w + (N\ \text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \leq 0.$$

Proof: We know $z_1 \geq 0$, $z_1^{'} > 0$, $z_2 \geq 0$ and $z_2^{'} < 0$. So $z_2^{'}z_1 - z_2z_1^{'} \leq 0$. If $h(\xi) \leq 0$ then

$$\int_a^{\xi} \frac{z_1(s)h(s)}{(z_2^{'}z_1 - z_2z_1^{'})}ds \geq 0 \quad \text{and} \quad \int_a^{\xi} \frac{z_2(s)h(s)}{(z_2^{'}z_1 - z_2z_1^{'})}ds \geq 0.$$

Also

$$\int_{\xi}^{b} \frac{z_1(s)h(s)}{(z_2^{'}z_1 - z_2z_1^{'})}ds \geq 0 \quad \text{and} \quad \int_{\xi}^{b} \frac{z_2(s)h(s)}{(z_2^{'}z_1 - z_2z_1^{'})}ds \geq 0.$$

We are given that

$$w(\xi) = z_2(\xi)\Big[\int_a^{\xi} \frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] + z_1(\xi)\Big[\int_{\xi}^{b} \frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] + \frac{Az_2(\xi)}{z_2(a)} + \frac{Bz_1(\xi)}{z_1(b)}.$$

From the proof of Lemma3.2.9we have

$$w^{'}(\xi) = z_2^{'}(\xi)\Big[\int_a^{\xi} \frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] + z_1^{'}(\xi)\Big[\int_{\xi}^{b} \frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s) - z_2(s)z_1^{'}(s)}ds\Big] + \frac{Az_2^{'}(\xi)}{z_2(a)} + \frac{Bz_1^{'}(\xi)}{z_1(b)}.$$

Now

$$(M-l)w+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))w^{'}$$
$$=(M-l)z_2(\xi)\Big[\int\limits_a^{\xi}\frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]$$
$$+(M-l)z_1(\xi)\Big[\int\limits_{\xi}^{b}\frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]+(M-l)\frac{Az_2(\xi)}{z_2(a)}+(M-l)\frac{Bz_1(\xi)}{z_1(b)}$$
$$+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))z_2^{'}(\xi)\Big[\int\limits_a^{\xi}\frac{z_1(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]$$
$$+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))z_1^{'}(\xi)\Big[\int\limits_{\xi}^{b}\frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]$$
$$+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))\frac{Az_2^{'}(\xi)}{z_2(a)}+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))\frac{Bz_1^{'}(\xi)}{z_1(b)},$$
$$=((M-l)z_2(\xi)+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))z_2^{'}(\xi))\Big[\int\limits_{\xi}^{b}\frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]$$
$$+((M-l)z_1(\xi)+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))z_1^{'}(\xi))\Big[\int\limits_{\xi}^{b}\frac{z_2(s)h(s)}{z_2^{'}(s)z_1(s)-z_2(s)z_1^{'}(s)}ds\Big]$$
$$+((M-l)z_2(\xi)+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))z_2^{'}(\xi))\frac{A}{z_2(a)}$$
$$+((M-l)z_1(\xi)+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))z_1^{'}(\xi))\frac{B}{z_1(b)}\leq 0.$$

The last inequality follows from the inequalities in Lemma3.2.12. ∎

In the following lemma we will prove that the derivative of the subsolution is bounded which will be needed to prove the main theorem.

Lemma 3.2.14 *Let* $\underline{\alpha}$ *and* $\bar{\beta}$ *satisfy* **[C1]** *and* f *satisfies* **[C2]** *and* **[C3]**. *Then for all* $C \geq 0$ *there exists* $R > 0$ *such that any solution* u *of*

$$u'' \geq f(\xi, u, u'), \qquad |u'(a)| \leq C, \quad |u'(b)| \leq C, \tag{3.32}$$

with $\underline{\alpha} \leq u \leq \bar{\beta}$*, satisfies* $|u'|_\infty < R$*.*

Proof: We know from [**C3**]

$$| f(\xi, u, u_1') - f(\xi, u, u_2') | \leq N \mid u_1' - u_2' \mid . \tag{3.33}$$

Choosing $u_2' = 0$ and $u_1' = u'$ then we have

$$| f(\xi, u, u') - f(\xi, u, 0) | \leq N \mid u' \mid,$$

which implies

$$-N \mid u' \mid \leq f(\xi, u, u') - f(\xi, u, 0) \leq N \mid u' \mid .$$

We will only consider left inequality,

$$f(\xi, u, u') - f(\xi, u, 0) \geq -N \mid u' \mid .$$

Adding $f(\xi, u, 0)$ on the both sides of the above inequality we have

$$f(\xi, u, u') \geq f(\xi, u, 0) - N \mid u' \mid .$$

Since $f(\xi, u, 0)$ is continuous and defined on a closed bounded domain, $f(\xi, u, 0)$ is bounded. Hence we can find a $\theta > 0$ so that $f(\xi, u, 0) \geq -\theta$, then we have from the above inequality

$$f(\xi, u, u') \geq -\theta - N \mid u' \mid . \tag{3.34}$$

Assume by contradiction that there exists a $C \geq 0$ such that for all $R > C$, there exist a function u which satisfies (3.32) and there exist $\bar{\xi} \in [a, b]$ so that $u'(\bar{\xi}) \geq R$. If

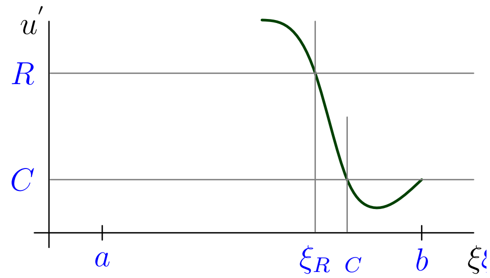


Figure 3.4: $u^{'}$ not bounded above by R.

so there exist a $\xi_R \in (a, b)$ such that $u^{'}(\xi_R) = R$. As $u^{'}(b) \leq C$, there exists $\xi_C > \xi_R$ so that $u^{'}(\xi_C) = C$ and $C < u^{'}(\xi) < R$ for all $\xi \in (\xi_R, \xi_C)$ as shown in the Figure3.4. Now from (3.32) we have

$$u^{''} \geq f(\xi, u, u^{'}),$$

which with the help of (3.34) we can write as

$$u^{''} \geq -\theta - N|u^{'}|.$$

Dividing both sides of the inequality by $-\theta - N|u^{'}|$, which is negative, and rearranging we have

$$1 \geq \frac{u^{''}}{-\theta - N \mid u^{'} \mid}.$$

Integrating the last inequality from ξ_R to ξ_C, we get

$$b - a \geq \xi_C - \xi_R \geq \int\limits_{\xi_R}^{\xi_C} \frac{u^{''}}{-\theta - N \mid u^{'} \mid} d\xi.$$

Now let $u^{'}(\xi) = s$ and substituting the limits of integral, we get

$$b - a \geq \int_C^R \frac{1}{\theta + Ns} ds,$$

that is

$$b - a \geq \frac{1}{N}(\ln(\theta + NR) - \ln(\theta + NC)).$$

This is a contradiction, as we can choose R as big as we wish.

Again assume by contradiction that there exists a $C \geq 0$ such that for all $R > C$, there exist a function u which satisfies (3.32) and there exist $\bar{\xi} \in [a, b]$ so that $u^{'}(\bar{\xi}) \leq -R$. Then there exist a $\xi_R \in (a, b)$ such that $u^{'}(\xi_R) = -R$. As $u^{'}(b) \geq -C$, there exists $\xi_C > \xi_R$ so that $u^{'}(\xi_C) = -C$ and $-R < u^{'}(\xi) < -C$ for all $\xi \in (\xi_R, \xi_C)$ as shown in Figure3.5. From (3.32) we have

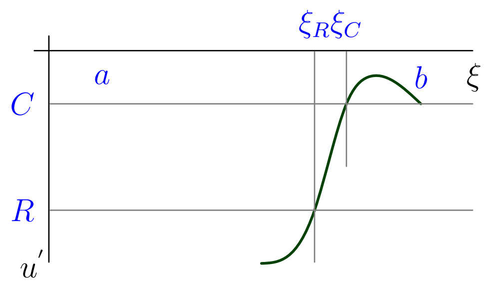


Figure 3.5: $u^{'}$ not bounded below by $-R$.

$$u^{''} \geq f(\xi, u, u^{'}),$$

which with the help of (3.34) we can write as

$$u^{''} \geq -\theta - N|u^{'}|.$$

Dividing both sides of the inequality by $-\theta - N|u^{'}|$ which is negative and rearranging we have

$$1 \geq \frac{u^{''}}{-\theta - N \mid u^{'} \mid}.$$

Integrating the last inequality from ξ_R to ξ_C, we get

$$b - a \geq \xi_C - \xi_R \geq \int_{\xi_R}^{\xi_C} \frac{u^{''}}{-\theta - N \mid u^{'} \mid} d\xi.$$

Now let $u^{'}(\xi) = s$ and substituting the limits of integral, we get

$$\begin{aligned} b - a &\geq \int_{-C}^{-R} \frac{1}{\theta + Ns} ds, \\ &= \frac{1}{N}(\ln(\theta - NR) - \ln(\theta - NC)), \\ &= \frac{1}{N} \ln\left(\frac{\theta - NR}{\theta - NC}\right), \\ &= \frac{1}{N} \ln\left(\frac{NR - \theta}{NC - \theta}\right). \end{aligned}$$

Which is a contradiction, as we can choose R as big as we wish.

Hence we have proved that for all $C \geq 0$ there exists $R > 0$ so that any solution u of (3.32) with $\underline{\alpha} \leq u \leq \bar{\beta}$ satisfies $|u^{'}|_\infty < R$. ■

Now we will prove the derivative of supersolution is bounded which will be needed to prove the main theorem.

Lemma 3.2.15 *Let $\underline{\alpha}$ and $\bar{\beta}$ satisfy* **[C1]** *and f satisfies* **[C2]** *and* **[C3]**. *Then for all $C \geq 0$ there exists $R > 0$ such that any solution u of*

$$u^{''} \leq f(\xi, u, u^{'}), \qquad |u^{'}(a)| \leq C, \quad |u^{'}(b)| \leq C, \tag{3.35}$$

with $\underline{\alpha} \leq u \leq \bar{\beta}$, satisfies $|u^{'}|_\infty < R$.

Proof: We know from **[C3]**

$$| f(\xi, u, u_1^{'}) - f(\xi, u, u_2^{'}) | \leq N | u_1^{'} - u_2^{'} | . \tag{3.36}$$

Choosing $u_2^{'} = 0$ and $u_1^{'} = u^{'}$ then we have

$$| f(\xi, u, u^{'}) - f(\xi, u, 0) | \leq N | u^{'} |,$$

which implies

$$-N \mid u^{'} \mid \leq f(\xi, u, u^{'}) - f(\xi, u, 0) \leq N \mid u^{'} \mid .$$

We will only consider right inequality,

$$f(\xi, u, u^{'}) - f(\xi, u, 0) \leq N \mid u^{'} \mid .$$

Adding $f(\xi, u, 0)$ on the both sides of the above inequality we have

$$f(\xi, u, u^{'}) \leq f(\xi, u, 0) + N \mid u^{'} \mid .$$

Since $f(\xi, u, 0)$ is continuous and defined on a closed bounded domain, $f(\xi, u, 0)$ is bounded. Hence we can find a $\theta > 0$ so that $f(\xi, u, 0) \leq \theta$.

$$f(\xi, u, u^{'}) \leq \theta + N \mid u^{'} \mid . \tag{3.37}$$

Assume by contradiction that there exists a $C \geq 0$ such that for all $R > C$, there exist a function u which satisfies (3.32) and there exist $\bar{\xi} \in [a, b]$ so that $u^{'}(\bar{\xi}) \geq R$. Then there exist a $\xi_R \in (a, b)$ such that $u^{'}(\xi_R) = R$, as $u^{'}(a) \leq C$, there exists $\xi_C < \xi_R$ so that $u^{'}(\xi_C) = C$ and $C < u^{'}(\xi) < R$ for all $\xi \in (\xi_C, \xi_R)$ as shown in Figure3.6. Now

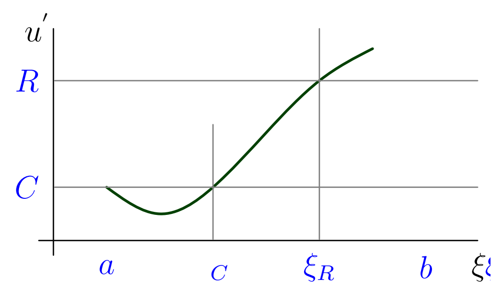


Figure 3.6: $u^{'}$ not bounded above by R.

from (3.35) we have

$$u^{''} \leq f(\xi, u, u^{'}),$$

which with the help of (3.37) we can write as

$$u^{''} \leq \theta + N|u^{'}|.$$

Dividing both sides of the inequality by the positive quantity $\theta + N|u^{'}|$ and rearranging we have

$$1 \geq \frac{u^{''}}{\theta + N \mid u^{'} \mid}.$$

Integrating the last inequality from ξ_C to ξ_R,

$$b - a \geq \xi_R - \xi_C \geq \int\limits_{\xi_C}^{\xi_R} \frac{u^{''}}{\theta + N \mid u^{'} \mid} d\xi.$$

Now let $u^{'}(\xi) = s$ and substituting the limits of integral, we get

$$b - a \geq \int_C^R \frac{1}{\theta + Ns} ds,$$

that is

$$b - a \geq \frac{1}{N}(\ln(\theta + NR) - \ln(\theta + NC)).$$

This is a contradiction, as we can choose R as big as we wish.

Again assume by contradiction that there exists a $C \geq 0$ such that for all $R > C$, there exist a function u which satisfies (3.32) and there exist $\bar{\xi} \in [a, b]$ so that $u^{'}(\bar{\xi}) \leq -R$. Then there exist a $\xi_R \in (a, b)$ such that $u^{'}(\xi_R) = -R$. As $u^{'}(a) \geq -C$, there exists $\xi_C < \xi_R$ so that $u^{'}(\xi_C) = -C$ and $-R < u^{'}(\xi) < -C$ for all $\xi \in (\xi_C, \xi_R)$ as shown in Figure3.7.

From (3.35) we have

$$u^{''} \leq f(\xi, u, u^{'}),$$

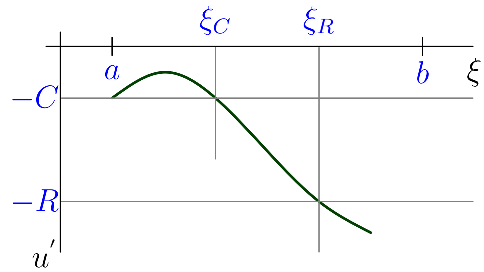


Figure 3.7: $u^{'}$ not bounded below by $-R$.

which with the help of (3.37) we can write as

$$u^{''} \leq \theta + N|u^{'}|.$$

Dividing both sides of the inequality by the positive quantity $\theta + N|u^{'}|$ and rearranging we have

$$1 \geq \frac{u^{''}}{\theta + N \mid u^{'} \mid}.$$

Integrating the last inequality from ξ_C to ξ_R, we get

$$b - a \geq \xi_R - \xi_C \geq \int_{\xi_C}^{\xi_R} \frac{u^{''}}{\theta + N \mid u^{'} \mid} d\xi.$$

Now let $u^{'}(\xi) = s$ and substituting the limits of integral, we get

$$\begin{aligned} b - a &\geq \int_{-C}^{-R} \frac{1}{\theta + Ns} ds, \\ &= \frac{1}{N}(\ln(\theta - NR) - \ln(\theta - NC)), \\ &= \frac{1}{N} \ln\left(\frac{\theta - NR}{\theta - NC}\right), \\ &= \frac{1}{N} \ln\left(\frac{NR - \theta}{NC - \theta}\right). \end{aligned}$$

This is a contradiction.

Hence we have proved that for all $C \geq 0$ there exists $R > 0$ so that any solution u of (3.32) with $\underline{\alpha} \leq u \leq \bar{\beta}$ satisfies $|u^{'}|_\infty < R$. ■

In the following lemma we will prove the monotonicity of the sequence if we start from a subsolution.

Lemma 3.2.16 *Consider $\alpha_0 = \underline{\alpha}$ (a subsolution of (3.17)) and for $n = 0, 1, 2, ...$ α_n defined by*

$$-\alpha_n^{''} + \sqrt[3]{l}K(\xi)\alpha_n^{'} + l\alpha_n = -f(\xi, \alpha_{n-1}, \alpha_{n-1}^{'}) + \sqrt[3]{l}K(\xi)\alpha_{n-1}^{'} + l\alpha_{n-1}, \tag{3.38}$$

$$\alpha_n(a) = 0, \qquad \alpha_n(b) = 0,$$

where $l > \max(M, \frac{N^3}{(K(a))^3})$ with $M, N \in \mathcal{R}^+$ and for all $\xi \in [a, b]$, $K(\xi) = -K(a+b-\xi)$ and $K(a) > 0$. Then for all $n \in \mathcal{N}$, α_n is a subsolution of the equation (3.17) and $\alpha_n \geq \alpha_{n-1}$.

Proof: By assumption $\alpha_0 = \underline{\alpha}$ is a subsolution of (3.17) and hence

$$h(\xi) = f(\xi, \alpha_0(\xi), \alpha_0^{'}(\xi)) - \alpha_0^{''} \leq 0.$$

Now rewriting equation (3.38) for $n = 1$ and adding $-\alpha_0^{''}$ to both sides we get

$$\alpha_1^{''} + \sqrt[3]{l}K(\xi)\alpha_1^{'} + l\alpha_1 - \alpha_0^{''} = -f(\xi, \alpha_0, \alpha_0^{'}) + \sqrt[3]{l}K(\xi)\alpha_0^{'} + l\alpha_0 - \alpha_0^{''}.$$

Rearranging we have

$$(\alpha_1^{''} - \alpha_0^{''}) - \sqrt[3]{l}K(\xi)(\alpha_1^{'} - \alpha_0^{'}) - l(\alpha_1 - \alpha_0) = f(\xi, \alpha_0, \alpha_0^{'}) - \alpha_0^{''},$$

that is

$$w^{''} - \sqrt[3]{l}K(\xi)w^{'} - lw \leq 0,$$

where $w = \alpha_1 - \alpha_0$. So w is a solution of (3.26) with $h(\xi) = f(\xi, \alpha_0(\xi), \alpha_0^{'}(\xi)) - \alpha_0^{''} \leq 0$. As $\alpha_1 = \alpha_0 = 0$ on $\partial\Omega$, we have $w = 0$ on $\partial\Omega$, hence we know from Lemma3.1.8that $w \geq 0$ in Ω, that is $\alpha_1 \geq \alpha_0$ in Ω.

We know from **[C3]**

$$| f(\xi, u, v_2) - f(\xi, u, v_1) | \leq N | v_2 - v_1 | .$$

Now considering $u = \alpha_1$, $v_2 = \alpha_1^{'}$ and $v_1 = \alpha_0^{'}$, we have from the above inequality

$$f(\xi, \alpha_1, \alpha_1^{'}) - f(\xi, \alpha_1, \alpha_0^{'}) \leq N | \alpha_1^{'} - \alpha_0^{'} |,$$

which we can write as

$$f(\xi, \alpha_1, \alpha_1^{'}) - f(\xi, \alpha_0, \alpha_0^{'}) + f(\xi, \alpha_0, \alpha_0^{'}) - f(\xi, \alpha_1, \alpha_0^{'}) \leq N | w^{'} | .$$

Adding $-(f(\xi, \alpha_0, \alpha_0^{'}) - f(\xi, \alpha_1, \alpha_0^{'}))$ on both sides of the inequality, we get

$$f(\xi, \alpha_1, \alpha_1^{'}) - f(\xi, \alpha_0, \alpha_0^{'}) \leq -f(\xi, \alpha_0, \alpha_0^{'}) + f(\xi, \alpha_1, \alpha_0^{'}) + N | w^{'} |,$$

and with the help of **[C2]** we can bound as

$$f(\xi, \alpha_1, \alpha_1^{'}) - f(\xi, \alpha_0, \alpha_0^{'}) \leq Mw + N | w^{'} |, \tag{3.39}$$

where $w = \alpha_1 - \alpha_0$. From equation (3.38), for $n = 1$, we get

$$-\alpha_1^{''} + \sqrt[3]{l}K(\xi)\alpha_1^{'} + l\alpha_1 = -f(\xi, \alpha_0, \alpha_0^{'}) + \sqrt[3]{l}K(\xi)\alpha_0^{'} + l\alpha_0.$$

Adding $f(\xi, \alpha_1, \alpha_1^{'}) - (\sqrt[3]{l}K(\xi)\alpha_1^{'} + l\alpha_1)$ on both sides of the above equation and rearranging, we have

$$f(\xi, \alpha_1, \alpha_1^{'}) - \alpha_1^{''} = f(\xi, \alpha_1, \alpha_1^{'}) - f(\xi, \alpha_0, \alpha_0^{'}) - \sqrt[3]{l}K(\xi)(\alpha_1^{'} - \alpha_0^{'}) - l(\alpha_1 - \alpha_0).$$

Using (3.39) we can obtain

$$f(\xi, \alpha_1, \alpha_1^{'}) - \alpha_1^{''} \leq (M-l)w + N \mid w^{'} \mid - \sqrt[3]{l}K(\xi)w^{'}.$$

As we know $\text{sign}(w^{'}) = \pm 1$, so $\mid w^{'} \mid = \text{sign}(w^{'})w^{'}$, and hence the above inequality can be written as

$$f(\xi, \alpha_1, \alpha_1^{'}) - \alpha_1^{''} \leq (M-l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'}.$$

Using Lemma3.2.13we know that

$$(M-l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \leq 0.$$

That implies

$$f(\xi, \alpha_1, \alpha_1^{'}) - \alpha_1^{''} \leq 0.$$

Hence we conclude that α_1 is a subsolution.

Assume further that our claim is true for some n and let us prove that it is true for $n+1$. Now rewrite equation (3.38) for $n+1$ and add $-\alpha_{(n)}^{''}$ on both sides to obtain

$$(\alpha_{(n+1)}^{''} - \alpha_{(n)}^{''}) - \sqrt[3]{l}K(\xi)(\alpha_{(n+1)}^{'} - \alpha_{(n)}^{'}) - l(\alpha_{(n+1)} - \alpha_{(n)}) = f(\xi, \alpha_{(n)}, \alpha_{(n)}^{'}) + \alpha_{(n)}^{''},$$

that is

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 = f(\xi, \alpha_{(n)}, \alpha_{(n)}^{'}) + \alpha_{(n)}^{''},$$

where $w_1 = \alpha_{n+1} - \alpha_n$. As our claim is true for n, that is α_n is a subsolution of (3.17), or $f(\xi, \alpha_{(n)}, \alpha_{(n)}^{'}) + \alpha_{(n)}^{''} \leq 0$, which implies

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq 0.$$

As $\alpha_{n+1} = \alpha_n = 0$ on $\partial\Omega$, or $w_1 = \alpha_{n+1} - \alpha_n = 0$ on $\partial\Omega$, hence by Lemma3.1.8 $w_1 \geq 0$ in Ω. Therefore we conclude that $\alpha_{n+1} \geq \alpha_n$ for all $n \in \mathcal{N}$.

We know from **[C3]**

$$| f(\xi, u, v_2) - f(\xi, u, v_1) | \leq N | v_2 - v_1 | .$$

Now choosing $u = \alpha_{n+1}$, $v_2 = \alpha'_{n+1}$ and $v_1 = \alpha'_n$ we have from the above inequality

$$f(\xi, \alpha_{n+1}, \alpha'_{n+1}) - f(\xi, \alpha_{n+1}, \alpha'_n) \leq N | \alpha'_{n+1} - \alpha'_n | .$$

Adding and subtracting $f(\xi, \alpha_n, \alpha'_n)$ we have

$$f(\xi, \alpha_{n+1}, \alpha'_{n+1}) - f(\xi, \alpha_n, \alpha'_n) + f(\xi, \alpha_n, \alpha'_n) - f(\xi, \alpha_{n+1}, \alpha'_n) \leq N | w'_1 |,$$

where again $w_1 = \alpha_{n+1} - \alpha_n$. Adding $-f(\xi, \alpha_n, \alpha'_n) + f(\xi, \alpha_{n+1}, \alpha'_n)$ on both sides of the above inequality, we have

$$f(\xi, \alpha_{n+1}, \alpha'_{n+1}) - f(\xi, \alpha_n, \alpha'_n) \leq -f(\xi, \alpha_n, \alpha'_n) + f(\xi, \alpha_{n+1}, \alpha'_n) + N | w'_1 | .$$

With the help of **[C2]** we can obtain the bound

$$f(\xi, \alpha_{n+1}, \alpha'_{n+1}) - f(\xi, \alpha_n, \alpha'_n) \leq M w_1 + N | w'_1 |, \tag{3.40}$$

From equation (3.38), for $n + 1$, we can write

$$-\alpha''_{n+1} + \sqrt[3]{l} K(\xi) \alpha'_{n+1} + l \alpha_{n+1} = -f(\xi, \alpha_n, \alpha'_n) + \sqrt[3]{l} K(\xi) \alpha'_n + l \alpha_n.$$

Adding $f(\xi, \alpha_{n+1}, \alpha'_{n+1}) - (\sqrt[3]{l} K(\xi) \alpha'_{n+1} + l \alpha_{n+1})$ on both sides of the above equation and rearranging, we have

$$\begin{aligned} f(\xi, \alpha_{n+1}, \alpha'_{n+1}) - \alpha''_{n+1} = & f(\xi, \alpha_{n+1}, \alpha'_{n+1}) - f(\xi, \alpha_n, \alpha'_n) \\ & - \sqrt[3]{l} K(\xi)(\alpha'_{n+1} - \alpha'_n) - l(\alpha_{n+1} - \alpha_n). \end{aligned}$$

As $w_1 = \alpha_{n+1} - \alpha_n$ we can write, with the help of (3.40),

$$f(\xi, \alpha_{n+1}, \alpha'_{n+1}) - \alpha''_{n+1} \leq (M-l)w_1 + N \mid w'_1 \mid - \sqrt[3]{l}K(\xi)w'_1.$$

Since $\mid w'_1 \mid = \mathrm{sign}(w'_1)w'_1$, the above inequality can be written as

$$f(\xi, \alpha_{n+1}, \alpha'_{n+1}) - \alpha''_{n+1} \leq (M-l)w_1 + (N\mathrm{sign}(w'_1) - \sqrt[3]{l}K(\xi))w'_1.$$

Hence by Lemma3.2.13we have

$$f(\xi, \alpha_{n+1}, \alpha'_{n+1}) - \alpha''_{n+1} \leq 0.$$

So α_{n+1} is a subsolution of (3.17). Therefore we conclude that for all n

$$f(\xi, \alpha_{(n+1)}, \alpha'_{(n+1)}) - \alpha''_{(n+1)} \leq 0,$$

that is for all n α_{n+1} is a subsolution of equation (3.17). ■

Lemma 3.2.17 *Consider $\beta_0 = \bar{\beta}$ (a supersolution of (3.17)) and for $n = 0, 1, 2, ...$ β_n defined by*

$$-\beta''_n + \sqrt[3]{l}K(\xi)\beta'_n + l\beta_n = -f(\xi, \beta_{n-1}, \beta'_{n-1}) + \sqrt[3]{l}K(\xi)\beta'_{n-1} + l\beta_{n-1}, \tag{3.41}$$
$$\beta_n(a) = 0, \qquad \beta_n(b) = 0,$$

where $l > \max(M, \frac{N^3}{(K(a))^3})$ with $M, N \in \mathcal{R}^+$ and for all $\xi \in [a, b]$, $K(\xi) = -K(a+b-\xi)$ and $K(a) > 0$. For all $n \in \mathcal{N}$, β_n is a subsolution of the equation (3.17) and $\beta_n \geq \beta_{n+1}$.

Proof: By assumption $\beta_0 = \bar{\beta}$ is a supersolution of (3.17) and so

$$h(\xi) = -f(\xi, \beta_0(\xi), \beta'_0(\xi)) + \beta''_0 \leq 0.$$

Now rewriting equation (3.41) for $n = 1$ and adding β''_0 to both sides we get

$$-\beta''_1 + \sqrt[3]{l}K(\xi)\beta'_1 + l\beta_1 + \beta''_0 = -f(\xi, \beta_0, \beta'_0) + \sqrt[3]{l}K(\xi)\beta'_0 + l\beta_0 + \beta''_0.$$

Rearranging we have

$$(\beta_0^{''} - \beta_1^{''}) - \sqrt[3]{l}K(\xi)(\beta_0^{'} - \beta_1^{'}) - l(\beta_0 - \beta_1) = -f(\xi, \beta_0, \beta_0^{'}) + \beta_0^{''},$$

that is

$$w^{''} - \sqrt[3]{l}K(\xi)w^{'} - lw \leq 0,$$

where $w = \beta_0 - \beta_1$. So w is a solution of (3.26) with $h(\xi) = -f(\xi, \beta_0(\xi), \beta_0^{'}(\xi)) + \beta_0^{''} \leq 0$. As $\beta_1 = \beta_0 = 0$ on $\partial\Omega$, or $w = 0$ on $\partial\Omega$, we know from Lemma3.1.8that $\quad w \geq 0$ in Ω or $\beta_0 \geq \beta_1$ in Ω.

We know from **[C3]**

$$\mid f(\xi, u, v_2) - f(\xi, u, v_1) \mid \leq N \mid v_2 - v_1 \mid .$$

Now considering $u = \beta_1$, $v_2 = \beta_1^{'}$ and $v_1 = \beta_0^{'}$ we have from the above inequality

$$N \mid \beta_1^{'} - \beta_0^{'} \mid \leq f(\xi, \beta_1, \beta_1^{'}) - f(\xi, \beta_1, \beta_0^{'}),$$

which we can write as

$$-N \mid w^{'} \mid \leq f(\xi, \beta_1, \beta_1^{'}) - f(\xi, \beta_0, \beta_0^{'}) + f(\xi, \beta_0, \beta_0^{'}) - f(\xi, \beta_1, \beta_0^{'}),$$

and with the help of **[C2]** we obtain

$$-N \mid w^{'} \mid \leq f(\xi, \beta_1, \beta_1^{'}) - f(\xi, \beta_0, \beta_0^{'}) + M(\beta_0 - \beta_1),$$

or

$$-N \mid w^{'} \mid \leq f(\xi, \beta_1, \beta_1^{'}) - f(\xi, \beta_0, \beta_0^{'}) + Mw.$$

Adding $-Mw$ on both sides and rearranging, we get

$$f(\xi, \beta_1, \beta_1^{'}) - f(\xi, \beta_0, \beta_0^{'}) \geq -N \mid w^{'} \mid -Mw, \tag{3.42}$$

where $w = \beta_0 - \beta_1$. From equation (3.41) for $n = 1$ we can write

$$- \beta_1^{''} + \sqrt[3]{l}K(\xi)\beta_1^{'} + l\beta_1 = -f(\xi, \beta_0, \beta_0^{'}) + \sqrt[3]{l}K(\xi)\beta_0^{'} + l\beta_0.$$

Now adding $f(\xi, \beta_1, \beta_1^{'}) - (\sqrt[3]{l}K(\xi)\beta_1^{'} + l\beta_1)$ on both sides and rearranging, we have

$$f(\xi, \beta_1, \beta_1^{'}) - \beta_1^{''} = f(\xi, \beta_1, \beta_1^{'}) - f(\xi, \beta_0, \beta_0^{'}) + \sqrt[3]{l}K(\xi)(\beta_0^{'} - \beta_1^{'}) + l(\beta_0 - \beta_1).$$

Using (3.42) we can obtain

$$-\beta_1^{''} + f(\xi, \beta_1, \beta_1^{'}) \geq -(M - l)w - Nw^{'} + \sqrt[3]{l}K(\xi)w^{'}.$$

Since $\mid w^{'} \mid = \text{sign}(w^{'})w^{'}$, the above inequality can be written as

$$-\beta_1^{''} + f(\xi, \beta_1, \beta_1^{'}) \geq -(M - l)w - (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'}.$$

So we only need to show

$$-(M - l)w - (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \geq 0,$$

that is

$$(M - l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \leq 0.$$

Using Lemma3.2.9and by Lemma3.2.13we know that

$$(M - l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \leq 0.$$

That implies

$$-\beta_1^{''} + f(\xi, \beta_1, \beta_1^{'}) \geq 0.$$

Hence we conclude that β_1 is a supersolution.

Assume further that our claim is true for some n and let us prove that it is true for $n+1$. Now rewrite equation (3.41) for $n+1$ and add $\beta_n^{''}$ on both sides to obtain

$$(\beta_n^{''} - \beta_{n+1}^{''}) - \sqrt[3]{l}K(\xi)(\beta_n^{'} - \beta_{n+1}^{'}) - l(\beta_n - \beta_{n+1}) = -f(\xi, \beta_n, \beta_n^{'}) + \beta_n^{''},$$

that is

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 = -f(\xi, \beta_n, \beta_n^{'}) + \beta_n^{''},$$

where $w_1 = \beta_n - \beta_{n+1}$. As our claim is true for n, that is β_n is a supersolution of (3.17), then $-f(\xi, \beta_n, \beta_n^{'}) + \beta_n^{''} \leq 0$, which implies

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq 0.$$

As $\beta_{n+1} = \beta_n = 0$ on $\partial\Omega$, or $w_1 = 0$ on $\partial\Omega$, by Lemma3.1.8 $w_1 \geq 0$ in Ω. Therefore we conclude that $\beta_n \geq \beta_{n+1}$ for all $n \in \mathcal{N}$.

We know from **[C3]**

$$\mid f(\xi, u, v_2) - f(\xi, u, v_1) \mid \leq N \mid v_2 - v_1 \mid .$$

Now choosing $u = \beta_{n+1}$, $v_2 = \beta_{n+1}^{'}$ and $v_1 = \beta_n^{'}$ we have from the above inequality

$$-N \mid \beta_{n+1}^{'} - \beta_n^{'} \mid \leq f(\xi, \beta_{n+1}, \beta_{n+1}^{'}) - f(\xi, \beta_{n+1}, \beta_n^{'}).$$

Adding and subtracting $f(\xi, \beta_n, \beta_n^{'})$ on right side of the above inequality we have

$$-N \mid w_1^{'} \mid \leq f(\xi, \beta_{n+1}, \beta_{n+1}^{'}) - f(\xi, \beta_n, \beta_n^{'}) + f(\xi, \beta_n, \beta_n^{'}) - f(\xi, \beta_{n+1}, \beta_n^{'}).$$

With the help of **[C2]** we can obtain

$$-N \mid w_1^{'} \mid \leq f(\xi, \beta_{n+1}, \beta_{n+1}^{'}) - f(\xi, \beta_n, \beta_n^{'}) + M(\beta_n - \beta_{n+1}).$$

Rearranging we have

$$f(\xi, \beta_{n+1}, \beta_{n+1}^{'}) - f(\xi, \beta_n, \beta_n^{'}) \geq -N \mid w_1^{'} \mid -Mw_1, \tag{3.43}$$

where again $w_1 = \beta_n - \beta_{n+1}$. From equation (3.41) for $n+1$ we can write,

$$-\beta_{n+1}^{''} + \sqrt[3]{l}K(\xi)\beta_{n+1}^{'} + l\beta_{n+1} = -f(\xi, \beta_n, \beta_n^{'}) + \sqrt[3]{l}K(\xi)\beta_n^{'} + l\beta_n.$$

Adding $f(\xi, \beta_{n+1}, \beta_{n+1}^{'}) - (\sqrt[3]{l}K(\xi)\beta_{n+1}^{'} + l\beta_{n+1})$ on both sides of the above equation and rearranging, we have

$$\begin{aligned} f(\xi, \beta_{n+1}, \beta_{n+1}^{'}) - \beta_{n+1}^{''} =& f(\xi, \beta_{n+1}, \beta_{n+1}^{'}) - f(\xi, \beta_n, \beta_n^{'}) \\ &+ \sqrt[3]{l}K(\xi)(\beta_n^{'} - \beta_{n+1}^{'}) + l(\beta_n - \beta_{n+1}). \end{aligned}$$

With $w_1 = \beta_n - \beta_{n+1}$ we can write this with the help of (3.43), as

$$f(\xi, \beta_{n+1}, \beta_{n+1}^{'}) - \beta_{n+1}^{''} \geq -(M-l)w_1 - Nw_1^{'} + \sqrt[3]{l}K(\xi))w_1^{'}.$$

Since $\mid w_1^{'} \mid = \text{sign}(w_1^{'})w_1^{'}$, the above inequality can be written as

$$f(\xi, \beta_{n+1}, \beta_{n+1}^{'}) - \beta_{n+1}^{''} \geq -(M-l)w_1 - (N\text{sign}(w_1^{'}) - \sqrt[3]{l}K(\xi))w_1^{'}.$$

So we only need to show that

$$-(M-l)w_1 - (N\text{sign}(w_1^{'}) - \sqrt[3]{l}K(\xi))w_1^{'} \geq 0,$$

that is

$$(M-l)w_1 + (N\text{sign}(w_1^{'}) - \sqrt[3]{l}K(\xi))w_1^{'} \leq 0.$$

By Lemma3.2.13we know that

$$(M-l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \leq 0.$$

So

$$f(\xi, \beta_{n+1}, \beta^{'}_{n+1}) - \beta^{''}_{n+1} \geq 0.$$

Hence for all n, $f(\xi, \beta_{n+1}, \beta^{'}_{n+1}) - \beta^{''}_{n+1} \geq 0$. That implies for all n, β_{n+1} is an supersolution of equation (3.17). ■

In the following lemma we will prove that the sequence $\{\alpha_n\}$ is bounded above by the supersolution $\bar{\beta}$ and the sequence $\{\beta_n\}$ is bounded below by the subsolution $\underline{\alpha}$.

Lemma 3.2.18 *Assume $\underline{\alpha}$ be a subsolution and $\bar{\beta}$ be a supersolution of equation (3.17) in Ω, where $l > \max(M, \frac{N^3}{(K(a))^3})$ with $M, N \in \mathcal{R}^+$ and for all $\xi \in [a, b]$, $K(\xi) = -K(a+b-\xi)$ and $K(a) > 0$. Consider the sequences α_n and β_n defined by, $\alpha_0 = \underline{\alpha}$ and for $n = 0, 1, 2, ...$*

$$-\alpha^{''}_{n+1} + \sqrt[3]{l}K(\xi)\alpha^{'}_{n+1} + l\alpha_{n+1} = -f(\xi, \alpha_n, \alpha^{'}_n) + \sqrt[3]{l}K(\xi)\alpha^{'}_n + l\alpha_n, \tag{3.44}$$
$$\alpha_{n+1}(a) = 0, \qquad \alpha_{n+1}(b) = 0,$$

and $\beta_0 = \bar{\beta}$ and for $n = 0, 1, 2, ...$

$$-\beta^{''}_{n+1} + \sqrt[3]{l}K(\xi)\beta^{'}_{n+1} + l\beta_{n+1} = -f(\xi, \beta_n, \beta^{'}_n) + \sqrt[3]{l}K(\xi)\beta^{'}_n + l\beta_n, \tag{3.45}$$
$$\beta_{n+1}(a) = 0, \qquad \beta_{n+1}(b) = 0,$$

respectively. Then we have

$$\underline{\alpha} \leq \alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n \leq \bar{\beta}.$$

Proof: We know from Lemma3.2.16and3.2.17that for all n, $\alpha_{n+1} \geq \underline{\alpha}$ and $\beta_{n+1} \leq \bar{\beta}$, so we only need to show that for all n, $\alpha_{n+1} \leq \beta_{n+1}$. Subtracting equation

(3.44) from equation (3.45) we get

$$-(\beta''_{n+1} - \alpha''_{n+1}) + \sqrt[3]{l}K(\xi)(\beta'_{n+1} - \alpha'_{n+1}) + l(\beta_{n+1} - \alpha_{n+1}) = -f(\xi, \beta_n, \beta'_n) + f(\xi, \alpha_n, \alpha'_n) + \sqrt[3]{l}K(\xi)(\beta'_n - \alpha'_n) + l(\beta_n - \alpha_n).$$

Let $w_1 = \beta_{n+1} - \alpha_{n+1}$ then we can rewrite the above equation as

$$-w''_1 + \sqrt[3]{l}K(\xi)w'_1 + lw_1 = -h(\xi),$$

or

$$w''_1 - \sqrt[3]{l}K(\xi)w'_1 - lw_1 = h(\xi), \tag{3.46}$$

where

$$h(\xi) = f(\xi, \beta_n, \beta'_n) - f(\xi, \alpha_n, \alpha'_n) - \sqrt[3]{l}K(\xi)(\beta'_n - \alpha'_n) - l(\beta_n - \alpha_n).$$

Let $w_2 = \beta_n - \alpha_n$ then we have

$$h(\xi) = f(\xi, \beta_n, \beta'_n) - f(\xi, \alpha_n, \beta'_n) + f(\xi, \alpha_n, \beta'_n) - f(\xi, \alpha_n, \alpha'_n) - \sqrt[3]{l}K(\xi)w'_2 - lw_2.$$

With the help of **[C2]** and **[C3]** we can obtain the bound

$$h(\xi) \leq M(\beta_n - \alpha_n) + N \mid \beta'_n - \alpha'_n \mid - \sqrt[3]{l}K(\xi)w'_2 - lw_2,$$

which implies

$$h(\xi) \leq (M - l)w_2 + (N\text{sign}(w'_2) - \sqrt[3]{l}K(\xi))w'_2.$$

By Lemma3.2.13we know $(M - l)w_2 + (N\text{sign}(w'_2) - \sqrt[3]{l}K(\xi))w'_2 \leq 0$, that is, for all n $h(\xi) \leq 0$. As $\beta_{n+1} = \alpha_{n+1} = 0$ on $\partial\Omega$ then $w_1 = \beta_{n+1} - \alpha_{n+1} = 0$ on $\partial\Omega$. So by Lemma3.1.8we conclude that $w_1 = \beta_{n+1} - \alpha_{n+1} \geq 0$ for all n, that is

$$\underline{\alpha} \leq \alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n \leq \bar{\beta}.$$

■

Lemma 3.2.19 *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be the sequences defined by* (3.19) *and* (3.20) *respectively on* $\Omega = [a, b]$. *Then there exists positive constants* μ_i , *for* $i = 1, 2, 3, 4$ *such that*

$$| \alpha_n^{'}(a) | \leq \mu_1, \qquad\qquad | \alpha_n^{'}(b) | \leq \mu_2,$$

and

$$| \beta_n^{'}(a) | \leq \mu_3, \qquad\qquad | \beta_n^{'}(b) | \leq \mu_4.$$

Proof: We know from Lemma3.2.18that on $\Omega = [\ \ a, b]$

$$\underline{\alpha} \leq \alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n \leq \bar{\beta}.$$

At point a,

$$\underline{\alpha}(a) = \alpha_n(a), \tag{3.47}$$

and at point $a + h$ with $h > 0$ and small

$$\underline{\alpha}(a + h) \leq \alpha_n(a + h). \tag{3.48}$$

Subtracting (3.47) from (3.48) and dividing both sides by h we have

$$\frac{\underline{\alpha}(a + h) - \underline{\alpha}(a)}{h} \leq \frac{\alpha_n(a + h) - \alpha_n(a)}{h}.$$

Now taking limit as $h \to 0$ on both sides we get

$$\underline{\alpha}^{'}(a) \leq \alpha_n^{'}(a).$$

Repeating procedure for supersolution we get

$$\alpha_n^{'}(a) \leq \bar{\beta}^{'}(a).$$

That is we have

$$\underline{\alpha}^{'}(a) \leq \alpha_n^{'}(a) \leq \bar{\beta}^{'}(a).$$

Now there arise four different cases. If $\underline{\alpha}^{'}(a)$ is positive then $\alpha_n^{'}(a)$ and $\bar{\beta}^{'}(a)$ are also positive. So in this case we have

$$-\bar{\beta}^{'}(a) \leq -\underline{\alpha}^{'}(a) \leq \underline{\alpha}^{'}(a) \leq \alpha_n^{'}(a) \leq \bar{\beta}^{'}(a),$$

which implies

$$\mid \alpha_n^{'}(a) \mid \leq \bar{\beta}^{'}(a).$$

If $\underline{\alpha}^{'}(a)$ is negative and $\alpha_n^{'}(a)$ and $\bar{\beta}^{'}(a)$ are positive then we have

$$-\bar{\beta}^{'}(a) \leq -\alpha_n^{'}(a) \leq \alpha_n^{'}(a) \leq \bar{\beta}^{'}(a),$$

which implies

$$\mid \alpha_n^{'}(a) \mid \leq \bar{\beta}^{'}(a). \tag{3.49}$$

In third case if we have $\bar{\beta}^{'}(a)$ is negative then $\underline{\alpha}^{'}(a)$ and $\alpha_n^{'}(a)$ are also negative and we have

$$\underline{\alpha}^{'}(a) \leq \alpha_n^{'}(a) \leq \bar{\beta}^{'}(a).$$

Multiplying by -1 we have

$$-\underline{\alpha}^{'}(a) \geq -\alpha_n^{'}(a) \geq -\bar{\beta}^{'}(a).$$

As $\alpha_n^{'}(a)$ is negative then $-\alpha_n^{'}(a)$ is positive and we have

$$\underline{\alpha}^{'}(a) \leq \alpha_n^{'}(a) \leq -\alpha_n^{'}(a) \leq -\underline{\alpha}^{'}(a),$$

or

$$| \alpha_n^{'}(a) | \leq -\underline{\alpha}^{'}(a).$$

Finally if $\bar{\beta}^{'}(a)$ is positive and $\underline{\alpha}^{'}(a)$ and $\alpha_n^{'}(a)$ are negative then we will also have

$$\underline{\alpha}^{'}(a) \leq \alpha_n^{'}(a) \leq -\alpha_n^{'}(a) \leq -\underline{\alpha}^{'}(a),$$

which implies

$$| \alpha_n^{'}(a) | \leq -\underline{\alpha}^{'}(a). \tag{3.50}$$

In fact we can see that

$$| \alpha_n^{'}(a) | \leq \max\{| \underline{\alpha}^{'}(a) |, | \bar{\beta}^{'}(a) |\}.$$

To see this, we know from inequality (3.49) that

$$| \alpha_n^{'}(a) | \leq \bar{\beta}^{'}(a),$$

we can relax this inequality further as

$$| \alpha_n^{'}(a) | \leq \bar{\beta}^{'}(a) \leq | \bar{\beta}^{'}(a) | .$$

From (3.50) we have

$$| \alpha_n^{'}(a) | \leq -\underline{\alpha}^{'}(a) \leq | \underline{\alpha}^{'}(a) | .$$

From above these inequalities we find

$$| \alpha_n^{'}(a) | \leq \max\{| \underline{\alpha}^{'}(a) |, | \bar{\beta}^{'}(a) |\}.$$

Hence we conclude that there exists a positive constant μ_1 such that

$$| \alpha_n^{'}(a) | \leq \mu_1.$$

Repeating same procedure for β_n at point a, we conclude that there exists a positive constant μ_2 such that $\mid \beta_n'(a) \mid \leq \mu_3$.

At point b,

$$\underline{\alpha}(b) = \alpha_n(b), \tag{3.51}$$

and at point $b - h$ with $h > 0$ and small

$$\underline{\alpha}(b-h) \leq \alpha_n(b-h).$$

Multiplying above inequality by -1, we have

$$-\underline{\alpha}(b-h) \geq -\alpha_n(b-h). \tag{3.52}$$

Adding (3.51) and (3.52) and dividing both sides by h we have

$$\frac{\underline{\alpha}(b) - \underline{\alpha}(b-h)}{h} \geq \frac{\alpha_n(b) - \alpha_n(b-h)}{h}.$$

Now taking the limit as $h \to 0$ on both sides we get

$$\underline{\alpha}'(b) \geq \alpha_n'(b).$$

Repeating procedure for supersolution at b we have

$$\alpha_n'(b) \geq \bar{\beta}'(b).$$

That is we have

$$\bar{\beta}'(b) \leq \alpha_n'(b) \leq \underline{\alpha}'(b).$$

Now there arise four different cases. If $\bar{\beta}'(b)$ is positive then $\alpha_n'(b)$ and $\underline{\alpha}'(b)$ are also positive. So in this case we have

$$-\underline{\alpha}'(b) \leq -\bar{\beta}'(b) \leq \bar{\beta}'(b) \leq \alpha_n'(b) \leq \underline{\alpha}'(b),$$

which implies

$$| \alpha_n^{'}(b) | \leq \underline{\alpha}^{'}(b).$$

If $\bar{\beta}^{'}(b)$ is negative and $\alpha_n^{'}(b)$ and $\underline{\alpha}^{'}(b)$ are positive then we have

$$-\underline{\alpha}^{'}(b) \leq -\alpha_n^{'}(b) \leq \alpha_n^{'}(b) \leq \underline{\alpha}^{'}(b),$$

which implies

$$| \alpha_n^{'}(b) | \leq \underline{\alpha}^{'}(b). \tag{3.53}$$

In third case if we have $\underline{\alpha}^{'}(b)$ is negative then $\bar{\beta}^{'}(b)$ and $\alpha_n^{'}(b)$ are also negative and we have

$$\bar{\beta}^{'}(b) \leq \alpha_n^{'}(b) \leq \underline{\alpha}^{'}(b).$$

Multiplying by -1 we have

$$-\bar{\beta}^{'}(b) \geq -\alpha_n^{'}(b) \geq -\underline{\alpha}^{'}(b).$$

As $\alpha_n^{'}(b)$ is negative then $-\alpha_n^{'}(b)$ is positive and we have

$$\bar{\beta}^{'}(b) \leq \alpha_n^{'}(b) \leq -\alpha_n^{'}(b) \leq -\bar{\beta}^{'}(b),$$

or

$$| \alpha_n^{'}(b) | \leq -\bar{\beta}^{'}(b).$$

Finally if $\underline{\alpha}^{'}(b)$ is positive and $\bar{\beta}^{'}(b)$ and $\alpha_n^{'}(b)$ are negative then we will also have

$$\bar{\beta}^{'}(b) \leq \alpha_n^{'}(b) \leq -\alpha_n^{'}(b) \leq -\bar{\beta}^{'}(b),$$

which implies

$$| \alpha_n^{'}(b) | \leq -\bar{\beta}^{'}(b). \tag{3.54}$$

In fact we obtain

$$| \alpha_n^{'}(b) | \leq \max\{| \underline{\alpha}^{'}(b) |, | \bar{\beta}^{'}(b) |\}.$$

To see this, we know from inequality (3.53) that

$$| \alpha_n^{'}(b) | \leq \underline{\alpha}^{'}(b).$$

We can relax this inequality further as

$$| \alpha_n^{'}(b) | \leq \underline{\alpha}^{'}(b) \leq | \underline{\alpha}^{'}(b) | .$$

From (3.54) we have

$$| \alpha_n^{'}(b) | \leq -\bar{\beta}^{'}(b) \leq | \bar{\beta}^{'}(b) | .$$

From above these inequalities we see

$$| \alpha_n^{'}(b) | \leq \max\{| \underline{\alpha}^{'}(b) |, | \bar{\beta}^{'}(b) |\}.$$

Hence we conclude that there exists a positive constant μ_3 such that

$$| \alpha_n^{'}(b) | \leq \mu_3.$$

Repeating same procedure for β_n at point b, we conclude that there exists a positive constant μ_4 such that $| \beta_n^{'}(b) | \leq \mu_4$. Hence the result is proved. ■

Solution of a nonhomogeneous differential equation can be written in terms of Green's function, which is stated in the lemma below.

Lemma 3.2.20 *Let $G(\xi, s)$ be the Green's function for the nonhomogeneous equation*

$$u^{''} - \sqrt[3]{l}K(\xi)u^{'} - lu = h(\xi). \tag{3.55}$$

Then the solution of nonhomogeneous equation (3.55) subject to homogeneous boundary conditions $u(a) = 0$ and $u(b) = 0$ can be written as,

$$u(\xi) = \int_a^b G(\xi, s)h(s)ds.$$

Proof: The proof of this Lemma can be found in [19] or [17]. ■

Lemma 3.2.21 *Let $\mathcal{X} = \mathcal{C}^1([a,b])$. Also let $\underline{\alpha}$ and $\bar{\beta} \in \mathcal{X}$ be a subsolution and a supersolution of (3.17) respectively with $\underline{\alpha} \leq \bar{\beta}$ and let*

$$\epsilon = \{u \in \mathcal{X} | \underline{\alpha} \leq u \leq \bar{\beta}\}.$$

Let $T : \epsilon \to \mathcal{X}$ be defined by

$$T(u(\xi)) = \int_a^b G(\xi,s)[f(s,u(s),u^{'}(s)) - \sqrt[3]{l}K(s)u^{'}(s) - lu(s)]ds, \tag{3.56}$$

where $G(\xi,s)$ is the Green's function of

$$u^{''} - \sqrt[3]{l}K(\xi)u^{'} - lu = h(\xi), \quad u(a) = 0, u(b) = 0.$$

Then the fixed points of T are the solutions of equation

$$u^{''} = f(\xi,u,u^{'}).$$

Proof: At the fixed point

$$T(u^*) = u^*,$$

that is

$$u^* = \int_a^b G(\xi,s)[f(s,u^*(s),(u^*)^{'}(s)) - \sqrt[3]{l}K(s)(u^*)^{'}(s) - lu^*(s)]ds,$$

which implies u^* solves

$$(u^*)^{''} - \sqrt[3]{l}K(\xi)(u^*)^{'} - lu^* = f(\xi,u^*,(u^*)^{'}) - \sqrt[3]{l}K(\xi)(u^*)^{'} - lu^*.$$

Adding $\sqrt[3]{l}K(\xi)(u^*)^{'} + lu^*$ on both sides of the above equation we obtain

$$(u^*)^{''} = f(\xi,u^*,(u^*)^{'}),$$

that completes the proof of the lemma. ■

Theorem 3.2.22 *Let $\mathcal{X} = \mathcal{C}^1([a,b])$, $\mathcal{Z} = \mathcal{C}([a,b])$. Let $\underline{\alpha}$ and $\bar{\beta} \in \mathcal{X}$ be a subsolution and a supersolution of (3.17) respectively with $\underline{\alpha} \leq \bar{\beta}$ and let*

$$\epsilon = \{u \in \mathcal{X} | \underline{\alpha} \leq u \leq \bar{\beta}\}.$$

Let $T : \epsilon \to \mathcal{X}$ be the continuous operator defined in (3.56). Then,

$$\underline{\alpha} \leq T(\underline{\alpha}), \qquad \bar{\beta} \geq T(\bar{\beta}).$$

Furthermore the sequences $\{\alpha_n\}$ defined by $\alpha_0 = \underline{\alpha}$, $\alpha_n = T(\alpha_{n-1})$ converges monotonically in $\mathcal{X}$ to a fixed point $u_{\min}$ of T such that for all n, $\alpha_n \leq u_{\min}$ and if u is any other fixed point in ϵ then $u_{\min} \leq u$.

Proof: We know from Lemma3.2.18that

$$\underline{\alpha} = \alpha_0 \leq T(\alpha_0) = \alpha_1 \leq T(\alpha_1) = \alpha_2 \leq ... \quad ... \leq T(\bar{\beta}) \leq \bar{\beta}.$$

Hence $\{\alpha_n\}$ is included in ϵ. As the sequence is monotone and bounded, the point wise limit exists:

$$\lim_{n\to\infty} \alpha_n(\xi) = u_{\min}(\xi).$$

By Lemma3.2.19we have

$$\mid \alpha_n^{'}(a) \mid \leq \mu_1,$$

and

$$\mid \alpha_n^{'}(b) \mid \leq \mu_2.$$

Hence by Lemma3.2.14we conclude that first derivative of the sequence is bounded. That is there exists $R \geq 0$ such that $\mid \alpha_n^{'} \mid_\infty < R$. This implies for all n,

$$\left| \frac{\alpha_n(\xi_1) - \alpha_n(\xi_2)}{\xi_1 - \xi_2} \right| < R,$$

as $\xi_1 \to \xi_2$, where $\xi_1, \xi_2 \in [a,b]$. This implies the sequence of functions $\{\alpha_n\}$ has a Lipschitz constant and hence the sequence is equicontinuous. Hence by Theorem 3.1.9there exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ which converges in $\mathcal{X}$ and therefore in $\mathcal{Z}$. By uniqueness of the limit and the monotonicity of the sequence $\{\alpha_n\}$ we conclude that $\alpha_n \to u_{\min}$ in $\mathcal{X}$. We also claim that $u_{\min}$ is a fixed point of T. We notice that

$$\lim_{n\to\infty} T(\alpha_n) = \lim_{n\to\infty} \alpha_{n+1} = u_{\min}.$$

As T is continuous, we deduce

$$u_{\min} = T(u_{\min}).$$

As u is a fixed point then $T(u) = u$, also we know that for all n, $\alpha_n \leq u$, we conclude by recurrence that $\alpha_n = T(\alpha_{n-1}) \leq T(u) = u$, so,

$$u_{\min} = \lim_{n\to\infty} \alpha_n \leq u.$$

Hence the theorem is proved. ■

Theorem 3.2.23 *Let $\mathcal{X} = \mathcal{C}^1([a,b])$, $\mathcal{Z} = \mathcal{C}([a,b])$. Let $\underline{\alpha}$ and $\bar{\beta} \in \mathcal{X}$ be a subsolution and a supersolution of* (3.17) *respectively with $\underline{\alpha} \leq \bar{\beta}$ and let*

$$\epsilon = \{u \in \mathcal{X} | \underline{\alpha} \leq u \leq \bar{\beta}\}.$$

Let $T : \epsilon \to \mathcal{X}$ be the continuous operator defined in (3.56)*. Then*

$$\underline{\alpha} \leq T(\underline{\alpha}), \qquad \bar{\beta} \geq T(\bar{\beta}).$$

Furthermore the sequences $\{\beta_n\}$ defined by $\beta_0 = \bar{\beta}$, $\beta_n = T(\beta_{n-1})$ converges monotonically in $\mathcal{X}$ to a fixed point $u_{\max}$ of T such that for all n $u_{\max} \leq \beta_n$ and if u is the fixed point in ϵ then $u \leq u_{\max}$.

Proof: We know from Lemma3.2.18that

$$\underline{\alpha} \leq T(\underline{\alpha}) \leq ... \quad ... \leq \beta_2 = T(\beta_1) \leq \beta_1 = T(\beta_0) \leq \beta_0 = \bar{\beta}.$$

Hence $\{\beta_n\}$ is included in ϵ. As the sequence is monotone and bounded, the point wise limit exists:

$$\lim_{n\to\infty} \beta_n(\xi) = u_{\max}(\xi),$$

By Lemma3.2.19we have

$$\mid \beta_n^{'}(a) \mid \leq \mu_3,$$

and

$$\mid \beta_n^{'}(b) \mid \leq \mu_4.$$

Hence by Lemma3.2.15we conclude that first derivative of the sequence is bounded. That is there exists $R \geq 0$ such that $\mid \beta_n^{'} \mid_\infty < R$. This implies for all n,

$$\left| \frac{\beta_n(\xi_1) - \beta_n(\xi_2)}{\xi_1 - \xi_2} \right| < R,$$

as $\xi_1 \to \xi_2$, where $\xi_1, \xi_2 \in [a, b]$. This implies the sequence of functions $\{\beta_n\}$ has a Lipschitz constant, hence the sequence is equicontinuous. Hence by Theorem3.1.9 there exists a subsequence $\{\beta_{n_k}\} \subset \{\beta_n\}$ which converges in $\mathcal{X}$ and therefore in $\mathcal{Z}$. By uniqueness of the limit and the monotonicity of the sequence $\{\beta_n\}$ we conclude that $\beta_n \to u_{\max}$ in $\mathcal{X}$. We also claim that $u_{\max}$ is a fixed point of T. We notice that

$$\lim_{n\to\infty} T(\beta_n) = \lim_{n\to\infty} \beta_{n+1} = u_{\max}.$$

As T is continuous, we deduce

$$u_{\max} = T(u_{\max}).$$

As u is a fixed point then $T(u) = u$, also we know that for all n $\beta_n \geq u$, we conclude by recurrence that $\beta_n = T(\beta_{n-1}) \geq T(u) = u$, so,

$$u_{\max} = \lim_{n \to \infty} \beta_n \geq u.$$

Hence the theorem is proved. ■

In next chapter we will discuss about S.H. Lui's DD iterations and give a new domain decomposition version of Cherpion's iteration.

Chapter 4

Linearized Domain Decomposition approaches

So far we have discussed the single domain linearized iterative approaches to solve nonlinear BVPs. In this chapter we will try to analyze linearized domain decomposition approaches for BVPs. A domain decomposition approach splits the whole domain into several subdomains, imposes transmission conditions on the interfaces and uses the single domain solver for each subdomain. Finally when each subdomain problem is solved, the approximation for the whole domain is constructed and the process iterates.

4.1 Linearized domain decomposition method to solve $u'' = f(\xi, u)$

We begin by summarizes the work of Lui from [14] which considers the PDE

$$-\Delta u = f(\xi, u) \text{ on } \Omega, \quad u = h \text{ on } \partial\Omega. \tag{4.1}$$

Suppose the domain is decomposed into two overlapping subdomains as shown in Figure4.1.

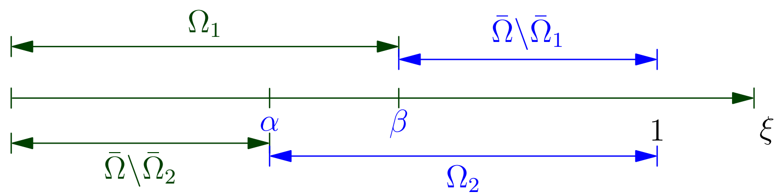


Figure 4.1: Domain decomposed into two subdomains.

Here $\bar{\Omega} \setminus \bar{\Omega}_1$ denotes the portion of the domain Ω_2 that does not overlap with Ω_1, $\bar{\Omega} \setminus \bar{\Omega}_2$ denotes the portion where Ω_1 does not overlap with Ω_2. $\bar{\Omega}$ is the closure of Ω.

Suppose $u^{(0)} = u^{(-\frac{1}{2})} = \underline{u}$ and define the alternating Schwarz sequence by $(n \geq 0)$,

$$-\Delta u^{(n+\frac{1}{2})} + cu^{(n+\frac{1}{2})} = f(u^{(n-\frac{1}{2})}) + cu^{(n-\frac{1}{2})} \text{ on } \Omega_1, \tag{4.2}$$
$$u^{(n+\frac{1}{2})} = u^{(n)} \text{ on } \partial\Omega_1,$$

and

$$-\Delta u^{(n+1)} + cu^{(n+1)} = f(u^{(n)}) + cu^{(n)} \text{ on } \Omega_2, \tag{4.3}$$
$$u^{(n+1)} = u^{(n+\frac{1}{2})} \text{ on } \partial\Omega_2.$$

Next we explain in detail a theorem which was stated and proved by S.H Lui in [14] for the BVP which has the form (4.1).

Theorem 4.1.1 *Let $u^{(0)} = u^{(-\frac{1}{2})} = \underline{u}$ on $\bar{\Omega}$ with $\underline{u} = h$ on $\partial\Omega$ and consider the iterations (4.2) and (4.3). Here $u^{(n+\frac{1}{2})}$ is defined as $u^{(n)}$ on $\bar{\Omega} \setminus \bar{\Omega}_1$ and $u^{(n+1)}$ is defined as $u^{(n+\frac{1}{2})}$ on $\bar{\Omega} \setminus \bar{\Omega}_2$. Then $u^{n+\frac{i}{2}} \to u$ in $C^2(\bar{\Omega})_i$ for $i = 1, 2$, where u is a solution of equation (4.1) in $\mathcal{A}$. If $u^{(0)} = u^{(-\frac{1}{2})} = \bar{u}$ on $\bar{\Omega}$ with $\bar{u} = h$ on $\partial\Omega$, then the same conclusion holds except that $u \geq v$ on $\bar{\Omega}$.*

Proof: First we need to demonstrate that the sequence is monotone and remains in $\mathcal{A}$, where $\mathcal{A}$ is the sector of smooth functions as defined in (3.4), that is we want to show

$$\underline{u} \leq u^{(n-\frac{1}{2})} \leq u^{(n)} \leq u^{(n+\frac{1}{2})} \leq \bar{u} \text{ on } \bar{\Omega}, n \geq 0. \tag{4.4}$$

Proof of (4.4):

To prove (4.4) we first show

$$u^{(n-\frac{1}{2})} \leq u^{(n+\frac{1}{2})} \text{ on } \bar{\Omega}_1 \quad \text{ and } \quad u^{(n)} \leq u^{(n+1)} \text{ on } \bar{\Omega}_2. \tag{4.5}$$

To do this we proceed by induction. On Ω_1, $u^{(-\frac{1}{2})} = u^{(0)}$, and for $n = 0$ we can write from (4.2)

$$-\triangle u^{(\frac{1}{2})} + cu^{(\frac{1}{2})} = f(u^{(0)}) + cu^{(0)}. \tag{4.6}$$

As $u^{(0)}$ is a subsolution, so

$$-\triangle u^{(0)} - f(u^{(0)}) \leq 0.$$

Adding $f(u^{(0)}) + cu^{(0)}$ on both sides of the above inequality, we get

$$-\triangle u^{(0)} + cu^{(0)} \leq f(u^{(0)}) + cu^{(0)}. \tag{4.7}$$

With the help of this inequality we get from (4.6)

$$-\triangle u^{(0)} + cu^{(0)} \leq -\triangle u^{(\frac{1}{2})} + cu^{(\frac{1}{2})}.$$

Rearranging above inequality we have

$$-\triangle(u^{(\frac{1}{2})} - u^{(0)}) + c(u^{(\frac{1}{2})} - u^{(0)}) \geq 0 \text{ on } \Omega_1.$$

Since $u^{(\frac{1}{2})} - u^{(0)} = 0$ on $\partial\Omega_1$ we conclude that $u^{(\frac{1}{2})} - u^{(0)} \geq 0$ on Ω_1 by Lemma3.1.8.

On Ω_2 for $n = 0$ we can write from (4.3)

$$-\triangle u^{(1)} + cu^{(1)} = f(u^{(0)}) + cu^{(0)}.$$

With the help of (4.7) we get from the above equation

$$-\triangle(u^{(1)} - u^{(0)}) + c(u^{(1)} - u^{(0)}) \geq 0 \text{ on } \Omega_2.$$

Since $u^{(1)} - u^{(0)} = 0$ on $\partial\Omega_2$ we conclude that $u^{(1)} - u^{(0)} \geq 0$ on Ω_2 by Lemma3.1.8. So we have shown that for $n = 0$, $u^{(-\frac{1}{2})} \leq u^{(\frac{1}{2})}$ on Ω_1 and $u^{(0)} \leq u^{(1)}$ on Ω_2.

Assume $u^{(n-\frac{1}{2})} \leq u^{(n+\frac{1}{2})}$ on Ω_1 and $u^{(n)} \leq u^{(n+1)}$ on Ω_2. We will prove this is true for $n+1$. Now subtracting the defining equations for $u^{(n+\frac{3}{2})}$ and $u^{(n+\frac{1}{2})}$ on Ω_1, we obtain

$$(-\triangle + c)(u^{(n+\frac{3}{2})} - u^{(n+\frac{1}{2})}) = f(u^{(n+\frac{1}{2})}) - f(u^{(n-\frac{1}{2})}) + c(u^{(n+\frac{1}{2})} - u^{(n-\frac{1}{2})}).$$

Since $u^{(n-\frac{1}{2})} \leq u^{(n+\frac{1}{2})}$ then by **[A1]** we know $-c(u^{(n+\frac{1}{2})} - u^{(n-\frac{1}{2})}) \leq f(u^{(n+\frac{1}{2})}) - f(u^{(n-\frac{1}{2})})$ which implies

$$(-\triangle + c)(u^{(n+\frac{3}{2})} - u^{(n+\frac{1}{2})}) \geq 0.$$

Now $u^{(n+\frac{3}{2})} = u^{(n+1)} \geq u^{(n)} = u^{(n+\frac{1}{2})}$ on $\partial\Omega_1$ so by Lemma3.1.8we conclude that $u^{(n+\frac{3}{2})} \geq u^{(n+\frac{1}{2})}$ on Ω_1.

Similarly subtracting the defining equations for $u^{(n+2)}$ and $u^{(n+1)}$ on Ω_2, we obtain

$$(-\triangle + c)(u^{(n+2)} - u^{(n+1)}) = f(u^{(n+1)}) - f(u^{(n)}) + c(u^{(n+1)} - u^{(n)}).$$

Since $u^{(n)} \leq u^{(n+1)}$ then by **[A1]** we know $-c(u^{(n+1)} - u^{(n)}) \leq f(u^{(n+1)}) - f(u^{(n)})$ which implies

$$(-\triangle + c)(u^{(n+2)} - u^{(n+1)}) \geq 0.$$

Furthermore $u^{(n+2)} = u^{(n+\frac{3}{2})} \geq u^{(n+\frac{1}{2})} = u^{(n+1)}$ on $\partial\Omega_2$ so by Lemma3.1.8we conclude that $u^{(n+2)} \geq u^{(n+1)}$ on Ω_2. So finally we have shown that (4.5) is true for all n. We will now use (4.5) to show that $u^{(n-\frac{1}{2})} \leq u^{(n)} \leq u^{(n+\frac{1}{2})}$ is true for all n.

On $\bar{\Omega}\backslash\bar{\Omega}_2$, $u^{(\frac{1}{2})} = u^{(1)}$ by definition. On $\Omega_{12} = \Omega_1 \cap \Omega_2$ subtract the defining equations for $u^{(\frac{1}{2})}$ and $u^{(1)}$ we obtain

$$(-\triangle + c)(u^{(1)} - u^{(\frac{1}{2})}) = f(u^{(0)}) - f(u^{(-\frac{1}{2})}) + c(u^{(0)} - u^{(-\frac{1}{2})}).$$

As $u^{(-\frac{1}{2})} = u^{(0)}$ that implies

$$(-\triangle + c)(u^{(1)} - u^{(\frac{1}{2})}) = 0.$$

On $\partial\Omega_1 \cap \Omega_2$ $u^{(1)} \geq u^{(\frac{1}{2})} = u^{(0)}$ and furthermore on $\partial\Omega_2 \cap \Omega_1$ $u^{(1)} = u^{(\frac{1}{2})}$ by the definition. Which implies that $u^{(1)} - u^{(\frac{1}{2})} \geq 0$ on $\partial\Omega_{12}$. Hence we conclude by Lemma 3.1.8that $u^{(1)} \geq u^{(\frac{1}{2})}$ on $\bar{\Omega}_{12}$. Since $u^{(\frac{1}{2})} = u^{(0)} \leq u^{(1)}$ on $\Omega_2\backslash\Omega_1$ then $u^{(1)} \geq u^{(\frac{1}{2})}$ on $\bar{\Omega}$.

On $\Omega_2\backslash\Omega_1$, $u^{(1)} = u^{(\frac{1}{2})}$ by definition. On $\Omega_{12} = \Omega_1 \cap \Omega_2$ the defining equations for $u^{(\frac{3}{2})}$ and $u^{(1)}$ give

$$-\triangle u^{(\frac{3}{2})} + cu^{(\frac{3}{2})} = f(u^{(\frac{1}{2})}) + cu^{(\frac{1}{2})}, \tag{4.8}$$

and

$$-\triangle u^{(1)} + cu^{(1)} = f(u^{(0)}) + cu^{(0)}. \tag{4.9}$$

Now subtracting (4.9) from (4.8), we obtain

$$(-\triangle + c)(u^{(\frac{3}{2})} - u^{(1)}) = f(u^{(\frac{1}{2})}) - f(u^{(0)}) + c(u^{(\frac{1}{2})} - u^{(0)}).$$

Since $u^{(\frac{1}{2})} \geq u^{(0)}$ then by [**A1**] the above equation implies

$$(-\triangle + c)(u^{(\frac{3}{2})} - u^{(1)}) \geq 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $u^{(\frac{3}{2})} \geq u^{(1)} = u^{(\frac{1}{2})}$ and on $\partial\Omega_2 \cap \Omega_1$, $u^{(1)} = u^{(\frac{1}{2})}$ by definition. Which implies that $u^{(\frac{3}{2})} - u^{(1)} \geq 0$ on $\partial\Omega_{12}$. Hence we conclude by Lemma3.1.8that $u^{(\frac{3}{2})} \geq u^{(1)}$ on $\bar{\Omega}_{12}$. Since $u^{(1)} = u^{(\frac{1}{2})} \leq u^{(\frac{3}{2})}$ on $\Omega_1 \setminus \Omega_2$ so $u^{(\frac{3}{2})} \geq u^{(1)}$ on $\bar{\Omega}$. That is (4.4) is true for $n = 1$.

Assume (4.4) is true for some n, now we will prove it is true for $n+1$. On $\Omega_1 \setminus \Omega_2$, $u^{(n+\frac{1}{2})} = u^{(n+1)}$ by definition. On $\Omega_{12} = \Omega_1 \cap \Omega_2$ subtract the defining equations for $u^{(n+\frac{1}{2})}$ and $u^{(n+1)}$ we obtain

$$(-\triangle + c)(u^{(n+1)} - u^{(n+\frac{1}{2})}) = f(u^{(n)}) - f(u^{(n-\frac{1}{2})}) + c(u^{(n)} - u^{(n-\frac{1}{2})}).$$

By [**A1**] the above equation implies

$$(-\triangle + c)(u^{(n+1)} - u^{(n+\frac{1}{2})}) \geq 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $u^{(n+1)} \geq u^{(n)} = u^{(n+\frac{1}{2})}$ while on $\partial\Omega_2 \cap \Omega_1$, $u^{(n+1)} = u^{(n+\frac{1}{2})}$ by definition. Which implies $u^{(n+1)} - u^{(n+\frac{1}{2})} \geq 0$ on $\partial\Omega_{12}$. Hence by Lemma3.1.8we conclude that $u^{(n+1)} \geq u^{(n+\frac{1}{2})}$ on $\bar{\Omega}_{12}$. Since $u^{(n+\frac{1}{2})} = u^{(n)} \leq u^{(n+1)}$ on $\Omega_2 \setminus \Omega_1$, we conclude $u^{(n+1)} \geq u^{(n+\frac{1}{2})}$ on $\bar{\Omega}$.

On $\Omega_2 \setminus \Omega_1$, $u^{(n+1)} = u^{(n+\frac{1}{2})}$ by definition. On $\Omega_{12} = \Omega_1 \cap \Omega_2$ the defining equations for $u^{(n+\frac{3}{2})}$ and $u^{(n+1)}$ give

$$-\triangle u^{(n+\frac{3}{2})} + cu^{(n+\frac{3}{2})} = f(u^{(n+\frac{1}{2})}) + cu^{(n+\frac{1}{2})}, \tag{4.10}$$

and

$$-\triangle u^{(n+1)} + cu^{(n+1)} = f(u^{(n)}) + cu^{(n)}. \tag{4.11}$$

Now subtracting (4.11) from (4.10), we obtain

$$(-\triangle + c)(u^{(n+\frac{3}{2})} - u^{(n+1)}) = f(u^{(n+\frac{1}{2})}) - f(u^{(n)}) + c(u^{(n+\frac{1}{2})} - u^{(n)}).$$

By [**A1**] the above equation implies

$$(-\triangle + c)(u^{(n+\frac{3}{2})} - u^{(n+1)}) \geq 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $u^{(n+\frac{3}{2})} \geq u^{(n+\frac{3}{2})} = u^{(n+\frac{1}{2})}$ and on $\partial\Omega_2 \cap \Omega_1$, $u^{(n+1)} = u^{(n+\frac{1}{2})}$ by definition. Which implies $u^{(n+\frac{3}{2})} - u^{(n+1)} \geq 0$ on $\partial\Omega_{12}$. Hence by Lemma3.1.8, $u^{(n+\frac{3}{2})} \geq u^{(n+1)}$ on $\bar{\Omega}_{12}$. Since $u^{(n+1)} = u^{(n+\frac{1}{2})} \leq u^{(n+\frac{3}{2})}$ on $\Omega_1 \setminus \Omega_2$, $u^{(n+\frac{3}{2})} \geq u^{(n+1)}$ on $\bar{\Omega}$.

We have proved that $u^{(n+\frac{1}{2})}$ and $u^{(n)}$ are monotonically increasing from subsolution $\underline{u}$ on $\bar{\Omega}$, so these sequences are bounded below by $\underline{u}$ in $\bar{\Omega}$. Now we need to show that for all $n \in \mathcal{N}$, $u^{(n+\frac{1}{2})} \leq \bar{u}$. From (4.2) for $n = 0$, we have

$$-\triangle u^{(\frac{1}{2})} + cu^{(\frac{1}{2})} = f(u^{(0)}) + cu^{(0)}, \tag{4.12}$$

as $u^{(-\frac{1}{2})} = u^{(0)} = \underline{u}$. Also we know $\bar{u}$ is a supersolution, so

$$-\triangle \bar{u} - f(\bar{u}) \geq 0.$$

Adding $f(\bar{u}) + c\bar{u}$ on both sides of the above inequality, we get

$$-\triangle \bar{u} + c\bar{u} \geq f(\bar{u}) + c\bar{u}. \tag{4.13}$$

Now subtracting (4.12) from (4.13) we have

$$-\triangle(\bar{u} - u^{(\frac{1}{2})}) + c(\bar{u} - u^{(\frac{1}{2})}) \geq f(\bar{u}) - f(u^{(0)}) + c(\bar{u} - u^{(0)}). \tag{4.14}$$

By property [**A1**] we know $-c(\bar{u} - u^{(0)}) \leq f(\bar{u}) - f(u^{(0)})$, so (4.14) implies

$$-\triangle(\bar{u} - u^{(\frac{1}{2})}) + c(\bar{u} - u^{(\frac{1}{2})}) \geq 0.$$

As $\bar{u} \geq u^{(\frac{1}{2})} = h$ on $\partial\Omega$ so by Lemma3.1.8we conclude that $^{-}u \geq u^{(\frac{1}{2})}$ on $\bar{\Omega}$. That is for $n = 0$, $\bar{u} \geq u^{(n+\frac{1}{2})}$ on $\bar{\Omega}$.

Assume that for some n, $\bar{u} \geq u^{(n+\frac{1}{2})}$, we will prove it is true for $n+1$ that is $\bar{u} \geq u^{(n+\frac{3}{2})}$. Now using the definition of equation for $u^{(n+\frac{3}{2})}$ and subtracting it from (4.13) we get,

$$-\triangle(\bar{u} - u^{(n+\frac{3}{2})}) + c(\bar{u} - u^{(n+\frac{3}{2})}) \geq f(\bar{u}) - f(u^{(n+\frac{1}{2})}) + c(\bar{u} - u^{(n+\frac{1}{2})}). \qquad (4.15)$$

By property [**A1**] we know $-c(\bar{u} - u^{(n+\frac{1}{2})}) \leq f(\bar{u}) - f(u^{(n+\frac{1}{2})})$, so (4.15) implies

$$-\triangle(\bar{u} - u^{(n+\frac{3}{2})}) + c(\bar{u} - u^{(n+\frac{3}{2})}) \geq 0.$$

As $\bar{u} \geq u^{(n+\frac{3}{2})} = h$ on $\partial\Omega$ so by Lemma3.1.8we conclude that $^{-}u \geq u^{(n+\frac{3}{2})}$ on $\bar{\Omega}$.

This concludes the proof of (4.4)**.**

Since the sequences are bounded above and monotonic, the point wise limits exist on $\bar{\Omega}$.

$$\lim_{n\to\infty} u^{(n+\frac{1}{2})}(\xi) = u_1(\xi) \qquad \text{and} \qquad \lim_{n\to\infty} u^{(n)}(\xi) = u_2(\xi).$$

From the point wise convergence of $\{u^{(n+\frac{1}{2})}\}$ and the continuity property of f, the sequences $\{f(u^{(n+\frac{1}{2})})\}$ converges point wise to $f(u_1)$ as $n \to \infty$. Also we know $\underline{u}(\xi) \leq u^{(n+\frac{1}{2})}(\xi) \leq \bar{u}(\xi)$ for all $n \in \mathcal{N}$ and for all $\xi \in \bar{\Omega}$, where $\bar{\Omega}$ is bounded domain. So, there exist a maximum of $\bar{u}(\xi)$ say M_1 and a minimum of $\underline{u}(\xi)$ say M_2 on $\bar{\Omega}$ such that $M_2 \leq u^{(n)}(\xi) \leq M_1$ for all $n \in \mathcal{N}$. As f is Hölder continuous $\bar{\Omega}$ so f is continuous on $\bar{\Omega}_1$. Hence there exists a constant C_1 such that

$$\| f(u^{(n+\frac{1}{2})}(\xi)) \|_{\mathcal{L}_\infty} \leq C_1 \text{ for all } n \in \mathcal{N} \text{ and for all } \xi \in \bar{\Omega}_1.$$

Now

$$\| f(u^{(n+\frac{1}{2})}(\xi)) \|_{\mathcal{L}_p} = \Big(\int_{\Omega_1} | f(u^{(n+\frac{1}{2})}(\xi)) |^p \, d\xi \Big)^{\frac{1}{p}} \leq \Big(\| f(u^{(n+\frac{1}{2})}(\xi)) \|^p_{\mathcal{L}_\infty} | \Omega_1 | \Big)^{\frac{1}{p}},$$

that is $\| f(u^{(n+\frac{1}{2})}(\xi)) \|_{\mathcal{L}_p} \leq C_2$, where $C_2 = \Big(\| f(u^{(n+\frac{1}{2})}(\xi)) \|^p_{\mathcal{L}_\infty} | \Omega_1 | \Big)^{\frac{1}{p}}$ which is finite. So $\{f(u^{(n+\frac{1}{2})})\}$ is uniformly bounded in $\mathcal{L}_p(\Omega_1)$ for every $p \geq 1$, where $\mathcal{L}_p(\Omega_1)$ is the Banach space as stated in Lemma3.1.4. Hence by Lemma3.1.4we can say that $\{u^{(n+\frac{1}{2})}\}$ is uniformly bounded in $\mathcal{W}^2_p(\Omega_1)$. We choose $p > N$ so that $\mu = 1 - \frac{N}{p}$, then by Lemma3.1.5, $\{u^{(n+\frac{1}{2})}\}$ is uniformly bounded in $\mathcal{C}^{1+\mu}(\bar{\Omega}_1)$. Furthermore by Lemma 3.1.6, $\{f(u^{(n+\frac{1}{2})})\}$ is uniformly bounded in $\mathcal{C}^{\mu}(\bar{\Omega}_1)$. So we conclude by Lemma3.1.7 that $\{u^{(n+\frac{1}{2})}\}$ is uniformly bounded in $\mathcal{C}^{2+\mu}(\bar{\Omega}_1)$. Hence by Theorem3.1.9there exists a subsequence of $\{u^{(n+\frac{1}{2})}\}$ which converges and Lemma3.1.10and Remark1confirm that this subsequence converges in $\mathcal{C}^2(\bar{\Omega}_1)$ to u^*. However $\{u^{(n+\frac{1}{2})}\}$ converges point wise to u_1 and as the sequence is monotonic the whole sequence $\{u^{(n+\frac{1}{2})}\}$ converges in $\mathcal{C}^2(\bar{\Omega}_1)$ to u_1, so we have $u_1 = u^*$. Now $f(u^{(n+\frac{1}{2})}) \to f(u_1)$ as $n \to \infty$ which implies that u_1 is a solution of (4.1) on Ω_1. Similarly we conclude that sequence $\{u^{(n+1)}\}$ converges in $\mathcal{C}^2(\bar{\Omega}_2)$ to u_2 and u_2 is a solution of (4.1) on Ω_2. From (4.4) we conclude that as $n \to \infty$ $u_1 = u_2$ on $\bar{\Omega}$ and we define $u = u_1$. $\blacksquare$

4.2 Linearized domain decomposition method to solve $u^{''} = f(\xi, u, u^{'})$

In this section we provide a domain decomposition extension of Cherpion's single domain iteration as detailed in Section3.2. Once again we consider the Dirichlet problem

$$-u^{''} + f(\xi, u, u^{'}) = 0, \quad u(a) = u(b) = 0. \tag{4.16}$$

We begin with some lemmas needed to prove our main theorem.

Lemma 4.2.1 *Let* $\alpha_{(0)} = \alpha_{(-\frac{1}{2})} = \underline{\alpha}$ *on* Ω *where* $\underline{\alpha}$ *is the subsolution of equation* (4.16) *on* Ω. *Define sequences by: for* $n = 0, 1, 2, ...,$

$$-\alpha''_{(n+\frac{1}{2})} + \sqrt[3]{l}K(\xi)\alpha'_{(n+\frac{1}{2})} + l\alpha_{(n+\frac{1}{2})} = -f(\xi, \alpha_{(n-\frac{1}{2})}, \alpha'_{(n-\frac{1}{2})}) + \sqrt[3]{l}K(\xi)\alpha'_{(n-\frac{1}{2})} + l\alpha_{(n-\frac{1}{2})}, \text{ on } \Omega_1 = [a, t], \tag{4.17}$$

$$\alpha_{(n+\frac{1}{2})} = \alpha_{(n)} \text{ on } \partial\Omega_1 = \{t\},$$

$$-\alpha''_{(n+1)} + \sqrt[3]{l}K(\xi)\alpha'_{(n+1)} + l\alpha_{(n+1)} = -f(\xi, \alpha_{(n)}, \alpha'_{(n)}) + \sqrt[3]{l}K(\xi)\alpha'_{(n)} + l\alpha_{(n)}, \text{ on } \Omega_2 = [s, b], \tag{4.18}$$

$$\alpha_{(n+1)} = \alpha_{(n+\frac{1}{2})} \text{ on } \partial\Omega_2 = \{s\},$$

where $s < t$ *and* $\alpha_{(n+\frac{1}{2})}$ *is defined as* $\alpha_{(n)}$ *on* $\Omega \setminus \Omega_1$ *and* $\alpha_{(n+1)}$ *is defined as* $\alpha_{(n+\frac{1}{2})}$ *on* $\Omega \setminus \Omega_2$. *Then for all* n, $\alpha_{(n+\frac{1}{2})}$ *and* $\alpha_{(n+1)}$ *are subsolutions of* (4.16) *on* Ω_1 *and* Ω_2 *respectively and for all* $n \in \mathcal{N}$,$\alpha_n \leq \alpha_{n+1}$ *and* $\alpha_{n+\frac{1}{2}} \leq \alpha_{n+\frac{3}{2}}$.

Proof: We will prove this theorem by induction. On Ω_1, for $n = 0$, equation (4.17)

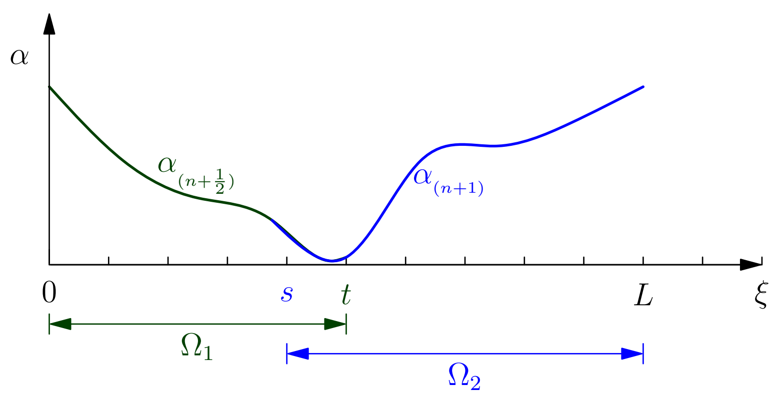


Figure 4.2: Iteration starting from the subsolution.

becomes

$$-\alpha''_{(\frac{1}{2})} + \sqrt[3]{l}K(\xi)\alpha'_{(\frac{1}{2})} + l\alpha_{(\frac{1}{2})} = - f(\xi, \alpha_{(-\frac{1}{2})}, \alpha'_{(-\frac{1}{2})}) + \sqrt[3]{l}K(\xi)\alpha'_{(-\frac{1}{2})} + l\alpha_{(-\frac{1}{2})}.$$

Since $\alpha_{(-\frac{1}{2})} = \alpha_{(0)}$, we have

$$-\alpha''_{(\frac{1}{2})} + \sqrt[3]{l}K(\xi)\alpha'_{(\frac{1}{2})} + l\alpha_{(\frac{1}{2})} = - f(\xi, \alpha_{(0)}, \alpha'_{(0)}) + \sqrt[3]{l}K(\xi)\alpha'_{(0)} + l\alpha_{(0)}. \tag{4.19}$$

Rearranging and adding $-\alpha''_{(0)}$ to both sides gives,

$$(\alpha''_{(\frac{1}{2})} - \alpha''_{(0)}) - \sqrt[3]{l}K(\xi)(\alpha'_{(\frac{1}{2})} - \alpha'_{(0)}) - l(\alpha_{(\frac{1}{2})} - \alpha_{(0)}) = f(\xi, \alpha_{(0)}, \alpha'_{(0)}) - \alpha''_{(0)},$$

or

$$w'' - \sqrt[3]{l}K(\xi)w' - lw = f(\xi, \alpha_{(0)}, \alpha'_{(0)}) - \alpha''_{(0)},$$

where $w = \alpha_{(\frac{1}{2})} - \alpha_{(0)}$. As $\alpha_{(0)}$ is a subsolution, we know $f(\xi, \alpha_{(0)}, \alpha'_{(0)}) - \alpha''_{(0)} \leq 0$ and $\alpha_{(\frac{1}{2})} = \alpha_{(0)}$ on $\partial\Omega_1$, that is $w = \alpha_{(\frac{1}{2})} - \alpha_{(0)} = 0$ on $\partial\Omega_1$. Hence from Lemma3.2.4 we conclude $w = \alpha_{(\frac{1}{2})} - \alpha_{(0)} \geq 0$, that is $\alpha_{(\frac{1}{2})} \geq \alpha_{(0)}$ on Ω_1.

We know from **[C3]**

$$| f(\xi, u, v_2) - f(\xi, u, v_1) | \leq N | v_2 - v_1 | . \tag{4.20}$$

Now considering $u = \alpha_{(\frac{1}{2})}$, $v_2 = \alpha'_{(\frac{1}{2})}$ and $v_1 = \alpha'_{(0)}$, we have from above inequality

$$f(\xi, \alpha_{(\frac{1}{2})}, \alpha'_{(\frac{1}{2})}) - f(\xi, \alpha_{(\frac{1}{2})}, \alpha'_{(0)}) \leq N | \alpha'_{(\frac{1}{2})} - \alpha'_{(0)} | .$$

We can rewrite this as

$$f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(\frac{1}{2})})-f(\xi,\alpha_{(0)},\alpha^{'}_{(0)})+f(\xi,\alpha_{(0)},\alpha^{'}_{(0)})-f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(0)})\leq N\mid w^{'}\mid,$$

where $w=\alpha_{(\frac{1}{2})}-\alpha_{(0)}$. Adding $-(f(\xi,\alpha_{(0)},\alpha^{'}_{(0)})-f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(0)}))$ on both sides of the inequality, we get

$$f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(\frac{1}{2})})-f(\xi,\alpha_{(0)},\alpha^{'}_{(0)})\leq -f(\xi,\alpha_{(0)},\alpha^{'}_{(0)})+f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(0)})+N\mid w^{'}\mid. \tag{4.21}$$

Since $\alpha_{(\frac{1}{2})}\geq\alpha_{(0)}$ then from **[C2]** we know $-f(\xi,\alpha_{(0)},\alpha^{'}_{(0)})+f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(0)})\leq M(\alpha_{(\frac{1}{2})}-\alpha_{(0)})$ and hence (4.21) becomes

$$f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(\frac{1}{2})})-f(\xi,\alpha_{(0)},\alpha^{'}_{(0)})\leq Mw+N\mid w^{'}\mid. \tag{4.22}$$

Adding $f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(\frac{1}{2})})$ on both sides of equation (4.19) and rearranging, we have

$$\begin{aligned}f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(\frac{1}{2})})-\alpha^{''}_{(\frac{1}{2})}=&f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(\frac{1}{2})})-f(\xi,\alpha_{(0)},\alpha^{'}_{(0)})-\\&\sqrt[3]{l}K(\xi)(\alpha^{'}_{(\frac{1}{2})}-\alpha^{'}_{(0)})-l(\alpha_{(\frac{1}{2})}-\alpha_{(0)}).\end{aligned}$$

With the help of (4.22) we find

$$f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(\frac{1}{2})})-\alpha^{''}_{(\frac{1}{2})}\leq(M-l)w+N\mid w^{'}\mid-\sqrt[3]{l}K(\xi)w^{'}.$$

As $\text{sign}(w^{'})=\pm1$ and $\mid w^{'}\mid=\text{sign}(w^{'})w^{'}$, the above inequality can be written as

$$f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(\frac{1}{2})})-\alpha^{''}_{(\frac{1}{2})}\leq(M-l)w+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))w^{'}.$$

For $l>\max(M,\frac{N^3}{(K(a))^3})$ we know from Lemma3.2.13

$$(M-l)w+(N\text{sign}(w^{'})-\sqrt[3]{l}K(\xi))w^{'}\leq 0.$$

That implies

$$f(\xi,\alpha_{(\frac{1}{2})},\alpha^{'}_{(\frac{1}{2})})-\alpha^{''}_{(\frac{1}{2})}\leq 0.$$

Hence we conclude that $\alpha_{(\frac{1}{2})}$ is a lower solution of (4.16).

Assume further that for some n, $\alpha_{(n+\frac{1}{2})}$ is a lower solution of (4.16), we will prove that $\alpha_{(n+\frac{3}{2})}$ is also a lower solution. Now evaluating (4.17) at iteration $n+1$

$$\begin{aligned} -\alpha''_{(n+\frac{3}{2})} + \sqrt[3]{l}K(\xi)\alpha'_{(n+\frac{3}{2})} + l\alpha_{(n+\frac{3}{2})} = & - f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) \\ & + \sqrt[3]{l}K(\xi)\alpha'_{(n+\frac{1}{2})} + l\alpha_{(n+\frac{1}{2})}, \end{aligned}$$

multiplying the above equation by -1 then adding $-\alpha''_{(n+\frac{1}{2})}$ on both sides and rearranging we get

$$\begin{aligned} & (\alpha''_{(n+\frac{3}{2})} - \alpha''_{(n+\frac{1}{2})}) - \sqrt[3]{l}K(\xi)(\alpha'_{(n+\frac{3}{2})} - \alpha'_{(n+\frac{1}{2})}) - l(\alpha_{(n+\frac{3}{2})} - \alpha_{(n+\frac{1}{2})}) \\ & = f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) - \alpha''_{(n+\frac{1}{2})}. \end{aligned}$$

This implies

$$w'' - \sqrt[3]{l}K(\xi)w' - lw = f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) - \alpha''_{(n+\frac{1}{2})},$$

where $w = \alpha_{(n+\frac{3}{2})} - \alpha_{(n+\frac{1}{2})}$. As $\alpha_{(n+\frac{1}{2})}$ is a lower solution we know $f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) - \alpha''_{(n+\frac{1}{2})} \leq 0$. By the construction of the iterations, $w \geq 0$ on $\partial\Omega_1$ and hence by Lemma 3.2.4 $w \geq 0$ on Ω_1 that is $\alpha_{(n+\frac{3}{2})} \geq \alpha_{(n+\frac{1}{2})}$ on Ω_1.

Now taking $u = \alpha_{(n+\frac{3}{2})}$, $v_2 = \alpha'_{(n+\frac{3}{2})}$ and $v_1 = \alpha'_{(n+\frac{1}{2})}$ we have from inequality (4.20)

$$f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha'_{(n+\frac{3}{2})}) - f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha'_{(n+\frac{1}{2})}) \leq N \mid \alpha'_{(n+\frac{3}{2})} - \alpha'_{(n+\frac{1}{2})} \mid .$$

We can rewrite this as

$$\begin{aligned} & f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha'_{(n+\frac{3}{2})}) - f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) + \\ & f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) - f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha'_{(n+\frac{1}{2})}) \leq N \mid w' \mid, \end{aligned}$$

where $w = \alpha_{(n+\frac{3}{2})} - \alpha_{(n+\frac{1}{2})}$. Adding $-(f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha^{'}_{(n+\frac{1}{2})}) - f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha^{'}_{(n+\frac{1}{2})}))$ on both sides of the above inequality we get

$$\begin{aligned} &f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha^{'}_{(n+\frac{3}{2})}) - f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha^{'}_{(n+\frac{1}{2})}) \leq \\ &- f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha^{'}_{(n+\frac{1}{2})}) + f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha^{'}_{(n+\frac{1}{2})}) + N \mid w^{'} \mid . \end{aligned}$$

With the help of **[C2]** we obtain

$$f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha^{'}_{(n+\frac{3}{2})}) - f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha^{'}_{(n+\frac{1}{2})}) \leq Mw + N \mid w^{'} \mid . \tag{4.23}$$

From equation (4.17) at iteration $n+1$ we have

$$\begin{aligned} -\alpha^{''}_{(n+\frac{3}{2})} + \sqrt[3]{l}K(\xi)\alpha^{'}_{(n+\frac{3}{2})} + l\alpha_{(n+\frac{3}{2})} = &- f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha^{'}_{(n+\frac{1}{2})}) + \\ &\sqrt[3]{l}K(\xi)\alpha^{'}_{(n+\frac{1}{2})} + l\alpha_{(n+\frac{1}{2})}. \end{aligned}$$

Adding $f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha^{'}_{(n+\frac{3}{2})}) - (\sqrt[3]{l}K(\xi)\alpha^{'}_{(n+\frac{3}{2})} + l\alpha_{(n+\frac{3}{2})})$ on both sides of the above equation and rearranging, we have

$$\begin{aligned} f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha^{'}_{(n+\frac{3}{2})}) - \alpha^{''}_{(n+\frac{3}{2})} = &f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha^{'}_{(n+\frac{3}{2})}) - f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha^{'}_{(n+\frac{1}{2})}) - \\ &\sqrt[3]{l}K(\xi)(\alpha^{'}_{(n+\frac{3}{2})} - \alpha^{'}_{(n+\frac{1}{2})}) - l(\alpha_{(n+\frac{3}{2})} - \alpha_{(n+\frac{1}{2})}). \end{aligned}$$

Using (4.23) we can write,

$$f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha^{'}_{(n+\frac{3}{2})}) - \alpha^{''}_{(n+\frac{3}{2})} \leq (M-l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'},$$

where $w = \alpha_{(n+\frac{3}{2})} - \alpha_{(n+\frac{1}{2})}$. Hence by Lemma3.2.13we know

$$(M-l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \leq 0.$$

So

$$f(\xi, \alpha_{(n+\frac{3}{2})}, \alpha^{'}_{(n+\frac{3}{2})}) - \alpha^{''}_{(n+\frac{3}{2})} \leq 0.$$

Hence for all n, $f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) - \alpha''_{(n+\frac{1}{2})} \leq 0$. This implies $\alpha_{(n+\frac{1}{2})}$ is a subsolution.

On Ω_2 for $n = 0$ equation (4.18) becomes,

$$\begin{aligned} -\alpha''_{(1)} + \sqrt[3]{l}K(\xi)\alpha'_{(1)} + l\alpha_{(1)} = &- f(\xi, \alpha_{(0)}, \alpha'_{(0)}) \\ &+ \sqrt[3]{l}K(\xi)\alpha'_{(0)} + l\alpha_{(0)}. \end{aligned}$$

Rearranging and adding $-\alpha''_{(0)}$ on both sides gives

$$(\alpha''_{(1)} - \alpha''_{(0)}) - \sqrt[3]{l}K(\xi)(\alpha'_{(1)} - \alpha'_{(0)}) - l(\alpha_{(1)} - \alpha_{(0)}) = f(\xi, \alpha_{(0)}, \alpha'_{(0)}) - \alpha''_{(0)}.$$

That is,

$$w'' - \sqrt[3]{l}K(\xi)w' - lw = f(\xi, \alpha_{(0)}, \alpha'_{(0)}) - \alpha''_{(0)},$$

where $w = \alpha_{(1)} - \alpha_{(0)}$. By assumption $\alpha_{(0)}$ is a subsolution of (4.16) and hence $f(\xi, \alpha_{(0)}, \alpha'_{(0)}) - \alpha''_{(0)} \leq 0$. As we have already seen that on Ω_1, $\alpha_{(\frac{1}{2})} \geq \alpha_{(0)}$ and $\partial\Omega_2$ lies in Ω_1 where $\alpha_1 = \alpha_{\frac{1}{2}}$. So $w = \alpha_{(1)} - \alpha_{(0)} \geq 0$ on $\partial\Omega_2$ and hence by Lemma3.2.4 $w \geq 0$ that is, $\alpha_{(1)} \geq \alpha_{(0)}$ on Ω_2.

Choosing $u = \alpha_{(1)}$, $v_2 = \alpha'_{(1)}$ and $v_1 = \alpha'_{(0)}$, we have from inequality (4.20)

$$f(\xi, \alpha_{(1)}, \alpha'_{(1)}) - f(\xi, \alpha_{(1)}, \alpha'_{(0)}) \leq N \mid \alpha'_{(1)} - \alpha'_{(0)} \mid .$$

Adding and subtracting $f(\xi, \alpha_{(0)}, \alpha'_{(0)})$ on the left side of the above inequality we have

$$f(\xi, \alpha_{(1)}, \alpha'_{(1)}) - f(\xi, \alpha_{(0)}, \alpha'_{(0)}) + f(\xi, \alpha_{(0)}, \alpha'_{(0)}) - f(\xi, \alpha_{(1)}, \alpha'_{(0)}) \leq N \mid w' \mid .$$

Adding $-(f(\xi, \alpha_{(0)}, \alpha'_{(0)}) - f(\xi, \alpha_{(1)}, \alpha'_{(0)}))$ on both sides we get

$$f(\xi, \alpha_{(1)}, \alpha'_{(1)}) - f(\xi, \alpha_{(0)}, \alpha'_{(0)}) \leq -f(\xi, \alpha_{(0)}, \alpha'_{(0)}) + f(\xi, \alpha_{(1)}, \alpha'_{(0)}) + N \mid w' \mid,$$

where $w = \alpha_{(1)} - \alpha_{(0)}$. With the help of **[C2]**, we have

$$f(\xi, \alpha_{(1)}, \alpha^{'}_{(1)}) - f(\xi, \alpha_{(0)}, \alpha^{'}_{(0)}) \leq Mw + N \mid w^{'} \mid . \tag{4.24}$$

From equation (4.18) for $n = 0$ we can write

$$-\alpha^{''}_{(1)} + \sqrt[3]{l}K(\xi)\alpha^{'}_{(1)} + l\alpha_{(1)} = - f(\xi, \alpha_{(0)}, \alpha^{'}_{(0)}) + \sqrt[3]{l}K(\xi)\alpha^{'}_{(0)} + l\alpha_{(0)}.$$

Adding $f(\xi, \alpha_{(1)}, \alpha^{'}_{(1)})$ on both sides and rearranging, we have

$$\begin{aligned} f(\xi, \alpha_{(1)}, \alpha^{'}_{(1)}) - \alpha^{''}_{(1)} =& f(\xi, \alpha_{(1)}, \alpha^{'}_{(1)}) - f(\xi, \alpha_{(0)}, \alpha^{'}_{(0)}) \\ & - \sqrt[3]{l}K(\xi)(\alpha^{'}_{(1)} - \alpha^{'}_{(0)}) - l(\alpha_{(1)} - \alpha_{(0)}), \end{aligned}$$

that is

$$f(\xi, \alpha_{(1)}, \alpha^{'}_{(1)}) - \alpha^{''}_{(1)} = f(\xi, \alpha_{(1)}, \alpha^{'}_{(1)}) - f(\xi, \alpha_{(0)}, \alpha^{'}_{(0)}) - \sqrt[3]{l}K(\xi)w^{'} - lw, \tag{4.25}$$

where $w = \alpha_{(1)} - \alpha_{(0)}$. With the help of (4.24), we get

$$f(\xi, \alpha_{(1)}, \alpha^{'}_{(1)}) - \alpha^{''}_{(1)} \leq (M - l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'}.$$

By Lemma3.2.13we know that

$$(M - l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \leq 0.$$

That implies

$$f(\xi, \alpha_{(1)}, \alpha^{'}_{(1)}) - \alpha^{''}_{(1)} \leq 0.$$

Hence we conclude that α_1 is a lower solution.

Assume further that for some n, $\alpha_{(n)}$ is a lower solution of (4.16), we will prove that $\alpha_{(n+1)}$ is also a lower solution. Now rearranging (4.18) and adding $-\alpha^{''}_{(n)}$ to both sides we get

$$(\alpha^{''}_{(n+1)} - \alpha^{''}_{(n)}) - \sqrt[3]{l}K(\xi)(\alpha^{'}_{(n+1)} - \alpha^{'}_{(n)}) - l(\alpha_{(n+1)} - \alpha_{(n)}) = f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) - \alpha^{''}_{(n)},$$

that is,

$$w^{''} - \sqrt[3]{l}K(\xi)w^{'} - lw = f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) - \alpha^{''}_{(n)},$$

where $w = \alpha_{(n+1)} - \alpha_{(n)}$. As $\alpha_{(n)}$ is a lower solution we know $f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) - \alpha^{''}_{(n)} \leq 0$ and hence

$$w^{''} - \sqrt[3]{l}K(\xi)w^{'} - lw \leq 0.$$

By the construction of iterations we have $w \geq 0$ on $\partial\Omega_2$, so by Lemma3.2.4 $w \geq 0$ on Ω_2, that is $\alpha_{(n+1)} \geq \alpha_{(n)}$ on Ω_2.

Now choosing $u = \alpha_{(n+1)}$, $v_2 = \alpha^{'}_{(n+1)}$ and $v_1 = \alpha^{'}_{(n)}$, we have from inequality (4.20)

$$f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n+1)}) - f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n)}) \leq N \mid \alpha^{'}_{(n+1)} - \alpha^{'}_{(n)} \mid .$$

Adding and subtracting $f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)})$ on the left side of the above inequality we have

$$f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n+1)}) - f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) + f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) - f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n)}) \leq N \mid w^{'} \mid .$$

Adding $-(f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) - f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n)}))$ on both sides we have

$$f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n+1)}) - f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) \leq -f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) + f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n)}) + N \mid w^{'} \mid .$$

With the help of **[C2]** we obtain the inequality

$$f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n+1)}) - f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) \leq Mw + N \mid w^{'} \mid, \tag{4.26}$$

where $w = \alpha_{(n+1)} - \alpha_{(n)}$. From equation (4.18) at iteration n we have

$$-\alpha^{''}_{(n+1)} + \sqrt[3]{l}K(\xi)\alpha^{'}_{(n+1)} + l\alpha_{(n+1)} = -f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) + \sqrt[3]{l}K(\xi)\alpha^{'}_{(n)} + l\alpha_{(n)}.$$

Adding $f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n+1)})$ on both sides of the above equation and rearranging, we have

$$
\begin{aligned}
f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n+1)}) - \alpha^{''}_{(n+1)} =& f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n+1)}) - f(\xi, \alpha_{(n)}, \alpha^{'}_{(n)}) \\
& - \sqrt[3]{l}K(\xi)(\alpha^{'}_{(n+1)} - \alpha^{'}_{(n)}) - l(\alpha_{(n+1)} - \alpha_{(n)}).
\end{aligned}
$$

With the help of (4.26) we get

$$f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n+1)}) - \alpha^{''}_{(n+1)} \leq (M - l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'}.$$

where $w = \alpha_{(n+1)} - \alpha_{(n)}$. By Lemma3.2.13we know that,

$$(M - l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \leq 0.$$

So

$$f(\xi, \alpha_{(n+1)}, \alpha^{'}_{(n+1)}) - \alpha^{''}_{(n+1)} \leq 0.$$

Hence for all n, α_{n+1} is a lower solution of (4.16). From the above we can conclude that the functions $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n+1)}$ defined by the iterates (4.17) and (4.18) are subsolutions of equation (4.16)) and these sequences are monotonically increasing. ■

In the following lemma we prove that iterates start from supersolution is decreasing monotonically.

Lemma 4.2.2 *Let $\beta_{(0)} = \beta_{(-\frac{1}{2})} = \bar{\beta}$ on Ω where $\bar{\beta}$ is the supersolution of equation (4.16) on Ω. Define sequences by: for $n = 0, 1, 2, ...$*

$$
\begin{aligned}
-\beta^{''}_{(n+\frac{1}{2})} + \sqrt[3]{l}K(\xi)\beta^{'}_{(n+\frac{1}{2})} + l\beta_{(n+\frac{1}{2})} =& -f(\xi, \beta_{(n-\frac{1}{2})}, \beta^{'}_{(n-\frac{1}{2})}) \\
& + \sqrt[3]{l}K(\xi)\beta^{'}_{(n-\frac{1}{2})} + l\beta_{(n-\frac{1}{2})} \text{ on } \Omega_1 = [0, t], \qquad (4.27) \\
\beta_{(n+\frac{1}{2})} = \beta_{(n)} & \text{ on } \partial\Omega_1 = \{t\},
\end{aligned}
$$

and

$$-\beta''_{(n+1)} + \sqrt[3]{l}K(\xi)\beta'_{(n+1)} + l\beta_{(n+1)} = -f(\xi, \beta_{(n)}, \beta'_{(n)}) + \sqrt[3]{l}K(\xi)\beta'_{(n)} + l\beta_{(n)} \text{ on } \Omega_2 = [s, b], \tag{4.28}$$

$$\beta_{(n+1)} = \beta_{(n+\frac{1}{2})} \text{ on } \partial\Omega_2 = \{s\},$$

where $s < t$ *and* $\beta_{(n+\frac{1}{2})}$ *is defined as* $\beta_{(n)}$ *on* $\Omega \setminus \Omega_1$ *and* $\beta_{(n+1)}$ *is defined as* $\beta_{(n+\frac{1}{2})}$ *on* $\Omega \setminus \Omega_2$*. Then for all* n*,* $\beta_{(n+\frac{1}{2})}$ *and* $\beta_{(n+1)}$ *are supersolutions of* (4.16) *on* Ω_1 *and* Ω_2 *respectively and for all* $n \in \mathcal{N}$*,* $\beta_{(n)} \geq \beta_{(n+1)}$ *and* $\beta_{(n+\frac{1}{2})} \geq \beta_{(n+\frac{3}{2})}$.

Proof: We will prove this by induction. On Ω_1 for $n = 0$ equation (4.27) becomes

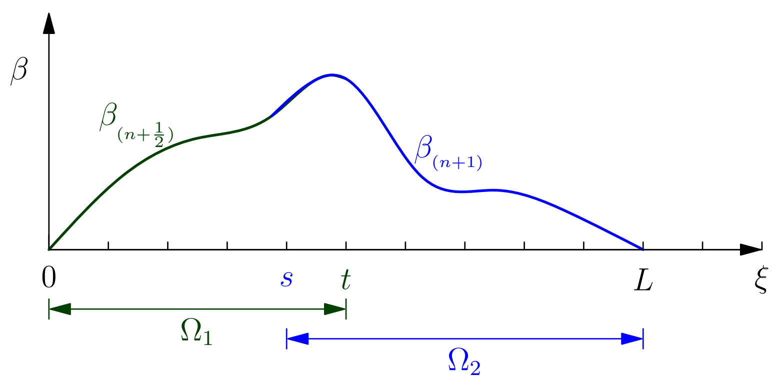


Figure 4.3: Iterations starting from the supersolution.

$$-\beta''_{(\frac{1}{2})} + \sqrt[3]{l}K(\xi)\beta'_{(\frac{1}{2})} + l\beta_{(\frac{1}{2})} = -f(\xi, \beta_{(-\frac{1}{2})}, \beta'_{(-\frac{1}{2})}) + \sqrt[3]{l}K(\xi)\beta'_{(-\frac{1}{2})} + l\beta_{(-\frac{1}{2})}.$$

Since $\beta_{(-\frac{1}{2})} = \beta_{(0)}$, we have

$$-\beta''_{(\frac{1}{2})} + \sqrt[3]{l}K(\xi)\beta'_{(\frac{1}{2})} + l\beta_{(\frac{1}{2})} = -f(\xi, \beta_{(0)}, \beta'_{(0)}) + \sqrt[3]{l}K(\xi)\beta'_{(0)} + l\beta_{(0)}.$$

Rearranging and adding β''_0 to both sides gives

$$(\beta''_{(0)} - \beta''_{(\frac{1}{2})}) - \sqrt[3]{l}K(\xi)(\beta'_{(0)} - \beta'_{(\frac{1}{2})}) - l(\beta_{(0)} - \beta_{(\frac{1}{2})}) = -f(\xi, \beta_{(0)}, \beta'_{(0)}) + \beta''_{(0)},$$

or

$$w^{''} - \sqrt[3]{l}K(\xi)w^{'} - lw = -f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) + \beta^{''}_{(0)},$$

where $w = \beta_{(0)} - \beta_{(\frac{1}{2})}$. As $\beta_{(0)}$ is a supersolution we know $f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) - \beta^{''}_{(0)} \geq 0$ that is $-f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) + \beta^{''}_{(0)} \leq 0$ and $w = \beta_{(0)} - \beta_{(\frac{1}{2})} = 0$ on $\partial\Omega_1$. Hence by Lemma 3.2.4we conclude $w = \beta_{(0)} - \beta_{(\frac{1}{2})} \geq 0$, that is $\beta_{(0)} \geq \beta_{(\frac{1}{2})}$ on Ω_1.

We know from **[C3]**

$$\mid f(\xi, u, v_2) - f(\xi, u, v_1) \mid \leq N \mid v_2 - v_1 \mid . \tag{4.29}$$

Considering $u = \beta_{(\frac{1}{2})}$, $v_2 = \beta^{'}_{(\frac{1}{2})}$ and $v_1 = \beta^{'}_{(0)}$, we have from the above inequality

$$-N \mid \beta^{'}_{(\frac{1}{2})} - \beta^{'}_{(0)} \mid \leq f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(\frac{1}{2})}) - f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(0)}).$$

We can rewrite this as

$$f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(\frac{1}{2})}) - f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) + f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) - f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(0)}) \geq -N \mid \beta^{'}_{(\frac{1}{2})} - \beta^{'}_{(0)} \mid .$$

With the help of **[C2]** we have

$$f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(\frac{1}{2})}) - f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) + M(\beta_{(0)} - \beta_{\frac{1}{2}}) \geq -N \mid \beta^{'}_{(\frac{1}{2})} - \beta^{'}_{(0)} \mid .$$

Let $w = \beta_{(0)} - \beta_{\frac{1}{2}}$. Now adding $-Mw$ on both sides of the inequality, we get

$$f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(\frac{1}{2})}) - f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) \geq -Mw - N \mid w^{'} \mid . \tag{4.30}$$

From equation (4.27) for $n = 0$ we can write

$$-\beta^{''}_{(\frac{1}{2})} + \sqrt[3]{l}K(\xi)\beta^{'}_{(\frac{1}{2})} + l\beta_{(\frac{1}{2})} = -f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) + \sqrt[3]{l}K(\xi)\beta^{'}_{(0)} + l\beta_{(0)}.$$

Adding $f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(\frac{1}{2})})$ on both sides of the above equation we get

$$\begin{aligned} f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(\frac{1}{2})}) - \beta^{''}_{(\frac{1}{2})} =& f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(\frac{1}{2})}) - f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) + \\ & \sqrt[3]{l}K(\xi)(\beta^{'}_{(0)} - \beta^{'}_{(\frac{1}{2})}) + l(\beta_{(0)} - \beta_{(\frac{1}{2})}). \end{aligned}$$

Since $| w^{'} |= \text{sign}(w^{'})w^{'}$, with the help of (4.30) the above inequality can be written as

$$f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(\frac{1}{2})}) - \beta^{''}_{(\frac{1}{2})} \geq -(M-l)w - (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'}.$$

By Lemma3.2.13we know

$$(M-l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \leq 0,$$

that is

$$-(M-l)w - (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \geq 0.$$

This implies

$$f(\xi, \beta_{(\frac{1}{2})}, \beta^{'}_{(\frac{1}{2})}) - \beta^{''}_{(\frac{1}{2})} \geq 0.$$

Hence we conclude that $\beta_{(\frac{1}{2})}$ is a supersolution of (4.16).

Assume further that for some n, $\beta_{(n+\frac{1}{2})}$ is an supersolution of (4.16), we will prove that $\beta_{(n+\frac{3}{2})}$ is also an supersolution. Now evaluating (4.27) at iteration $n+1$

$$-\beta^{''}_{(n+\frac{3}{2})} + \sqrt[3]{l}K(\xi)\beta^{'}_{(n+\frac{3}{2})} + l\beta_{(n+\frac{3}{2})} = -f(\xi, \beta_{(n+\frac{1}{2})}, \beta^{'}_{(n+\frac{1}{2})}) + \sqrt[3]{l}K(\xi)\beta^{'}_{(n+\frac{1}{2})} + l\beta_{(n+\frac{1}{2})}.$$

Adding $\beta^{''}_{(n+\frac{1}{2})}$ on both sides and rearranging, we get

$$\begin{aligned}(\beta^{''}_{(n+\frac{1}{2})} - \beta^{''}_{(n+\frac{3}{2})}) - \sqrt[3]{l}K(\xi)(\beta^{'}_{(n+\frac{1}{2})} - \beta^{'}_{(n+\frac{3}{2})}) - l(\beta_{(n+\frac{1}{2})} - \beta_{(n+\frac{3}{2})}) \\ = -f(\xi, \beta_{(n+\frac{1}{2})}, \beta^{'}_{(n+\frac{1}{2})}) + \beta^{''}_{(n+\frac{1}{2})},\end{aligned}$$

that is

$$w^{''} - \sqrt[3]{l}K(\xi)w^{'} - lw = -f(\xi, \beta_{(n+\frac{1}{2})}, \beta^{'}_{(n+\frac{1}{2})}) + \beta^{''}_{(n+\frac{1}{2})},$$

where $w = \beta_{(n+\frac{1}{2})} - \beta_{(n+\frac{3}{2})}$. As $\beta_{(n+\frac{1}{2})}$ is a supersolution, so $-f(\xi, \beta_{(n+\frac{1}{2})}, \beta^{'}_{(n+\frac{1}{2})}) + \beta^{''}_{(n+\frac{1}{2})} \leq 0$ which implies

$$w^{''} - \sqrt[3]{l}K(\xi)w^{'} - lw \leq 0.$$

By the construction of iterations $w \geq 0$ on $\partial\Omega_1$ and hence by Lemma3.2.4, $w \geq 0$ on Ω_1 that is $\beta_{(n+\frac{1}{2})} \geq \beta_{(n+\frac{3}{2})}$ on Ω_1.

Now taking $u = \beta_{(n+\frac{3}{2})}$, $v_2 = \beta^{'}_{(n+\frac{3}{2})}$ and $v_1 = \beta^{'}_{(n+\frac{1}{2})}$, we have from (4.29)

$$-N \mid \beta^{'}_{(n+\frac{3}{2}} - \beta^{'}_{(n+\frac{1}{2})} \mid \leq f(\xi, \beta_{(n+\frac{3}{2})}, \beta^{'}_{(n+\frac{3}{2})}) - f(\xi, \beta_{(n+\frac{3}{2})}, \beta^{'}_{(n+\frac{1}{2})}).$$

We can rewrite this as

$$\begin{aligned} f(\xi, \beta_{(n+\frac{3}{2})}, \beta^{'}_{(n+\frac{3}{2})}) - f(\xi, \beta_{(n+\frac{1}{2})}, \beta^{'}_{(n+\frac{1}{2})}) + f(\xi, \beta_{(n+\frac{1}{2})}, \beta^{'}_{(n+\frac{1}{2})}) \\ - f(\xi, \beta_{(n+\frac{3}{2})}, \beta^{'}_{(n+\frac{1}{2})}) \geq -N \mid w^{'} \mid, \end{aligned}$$

where $w = \beta_{(n+\frac{1}{2})} - \beta_{(n+\frac{3}{2})}$. With the help of **[C2]** we have

$$f(\xi, \beta_{(n+\frac{3}{2})}, \beta^{'}_{(n+\frac{3}{2})}) - f(\xi, \beta_{(n+\frac{1}{2})}, \beta^{'}_{(n+\frac{1}{2})}) + M(\beta_{(n+\frac{1}{2})} - \beta_{(n+\frac{3}{2})}) \geq -N \mid w^{'} \mid .$$

Adding $-Mw$ on both sides we obtain

$$f(\xi, \beta_{(n+\frac{3}{2})}, \beta^{'}_{(n+\frac{3}{2})}) - f(\xi, \beta_{(n+\frac{1}{2})}, \beta^{'}_{(n+\frac{1}{2})}) \geq -Mw - N \mid w^{'} \mid . \tag{4.31}$$

From equation (4.27) at iteration $n+1$ we obtain

$$\begin{aligned} -\beta^{''}_{(n+\frac{3}{2})} + \sqrt[3]{l}K(\xi)\beta^{'}_{(n+\frac{3}{2})} + l\beta_{(n+\frac{3}{2})} = - f(\xi, \beta_{(n+\frac{1}{2})}, \beta^{'}_{(n+\frac{1}{2})}) + \\ \sqrt[3]{l}K(\xi)\beta^{'}_{(n+\frac{1}{2})} + l\beta_{(n+\frac{1}{2})}. \end{aligned}$$

Adding $f(\xi, \beta_{(n+\frac{3}{2})}, \beta^{'}_{(n+\frac{3}{2})})$ on both sides

$$\begin{aligned} -\beta^{''}_{(n+\frac{3}{2})} + f(\xi, \beta_{(n+\frac{3}{2})}, \beta^{'}_{(n+\frac{3}{2})}) = f(\xi, \beta_{(n+\frac{3}{2})}, \beta^{'}_{(n+\frac{3}{2})}) - f(\xi, \beta_{(n+\frac{1}{2})}, \beta^{'}_{(n+\frac{1}{2})}) \\ + \sqrt[3]{l}K(\xi)(\beta^{'}_{(n+\frac{1}{2})} - \beta^{'}_{(n+\frac{3}{2})}) + l(\beta_{(n+\frac{1}{2})} - \beta_{(n+\frac{3}{2})}), \end{aligned}$$

that is

$$-\beta''_{(n+\frac{3}{2})} + f(\xi, \beta_{(n+\frac{3}{2})}, \beta'_{(n+\frac{3}{2})}) = f(\xi, \beta_{(n+\frac{3}{2})}, \beta'_{(n+\frac{3}{2})}) - f(\xi, \beta_{(n+\frac{1}{2})}, \beta'_{(n+\frac{1}{2})})$$
$$+ lw + \sqrt[3]{l}K(\xi)w',$$

where $w = \beta_{(n+\frac{1}{2})} - \beta_{(n+\frac{3}{2})}$. Using (4.31), we can write

$$-\beta_{(n+\frac{3}{2}'')} + f(\xi, \beta_{(n+\frac{3}{2})}, \beta_{(n+\frac{3}{2}')}) \geq -(M-l)w - (N\text{sign}(w') - \sqrt[3]{l}K(\xi))w'.$$

By Lemma3.2.13we know

$$-(M-l)w - (N\text{sign}(w') - \sqrt[3]{l}K(\xi))w' \geq 0,$$

that is

$$(M-l)w + (N\text{sign}(w') - \sqrt[3]{l}K(\xi))w' \leq 0.$$

This implies

$$-\beta''_{(n+\frac{3}{2})} + f(\xi, \beta_{(n+\frac{3}{2})}, \beta'_{(n+\frac{3}{2})}) \geq 0.$$

Hence for all n, $\beta_{(n+\frac{1}{2})}$ is a supersolution of (4.16).

On Ω_2 for $n = 0$ equation (4.28) becomes

$$-\beta''_{(1)} + \sqrt[3]{l}K(\xi)\beta'_{(1)} + l\beta_{(1)} = -f(\xi, \beta_{(0)}, \beta'_{(0)}) + \sqrt[3]{l}K(\xi)\beta'_{(0)} + l\beta_{(0)}.$$

Rearranging and adding β''_0 on both sides gives

$$(\beta''_{(0)} - \beta''_{(1)}) - \sqrt[3]{l}K(\xi)(\beta'_{(0)} - \beta'_{(1)}) - l(\beta_{(0)} - \beta_{(1)}) = -f(\xi, \beta_{(0)}, \beta'_{(0)}) + \beta''_{(0)},$$

that is

$$w'' - \sqrt[3]{l}K(\xi)w' - lw = -f(\xi, \beta_{(0)}, \beta'_{(0)}) + \beta''_{(0)},$$

where $w = \beta_{(0)} - \beta_{(1)}$. As $\beta_{(0)}$ is a supersolution we know $f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) - \beta^{''}_{(0)} \geq 0$ that is $-f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) + \beta^{''}_{(0)} \leq 0$ and we have already seen that $\beta_{(0)} \geq \beta_{(\frac{1}{2})}$ on Ω_1 and on $\partial\Omega_2$ $\beta_{(1)} = \beta_{(\frac{1}{2})}$. So on $\partial\Omega_2$ $\beta_{(0)} \geq \beta_{(1)}$. Hence $w \geq 0$ on $\partial\Omega_2$, then by Lemma 3.2.4, $w \geq 0$, that is, $\beta_{(0)} \geq \beta_{(1)}$ on Ω_2.

We know from **[C3]**

$$\mid f(\xi, u, v_2) - f(\xi, u, v_1) \mid \leq N \mid v_2 - v_1 \mid . \tag{4.32}$$

Choosing $u = \beta_{(1)}$, $v_2 = \beta^{'}_{(1)}$ and $v_1 = \beta^{'}_{(0)}$, we have from above inequality

$$-N \mid \beta^{'}_{(1)} - \beta^{'}_{(0)} \mid \leq f(\xi, \beta_{(1)}, \beta^{'}_{(1)}) - f(\xi, \beta_{(1)}, \beta^{'}_{(0)}).$$

Adding and subtracting $f(\xi, \beta_{(0)}, \beta^{'}_{(0)})$ on the left side we have

$$f(\xi, \beta_{(1)}, \beta^{'}_{(1)}) - f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) + f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) - f(\xi, \beta_{(1)}, \beta^{'}_{(0)}) \geq -N \mid \beta^{'}_{(1)} - \beta^{'}_{(0)} \mid .$$

With the help of **[C2]** we have

$$f(\xi, \beta_{(1)}, \beta^{'}_{(1)}) - f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) + M(\beta_{(0)} - \beta_{(1)}) \geq -N \mid \beta^{'}_{(1)} - \beta^{'}_{(0)} \mid .$$

Let $w = \beta_{(1)} - \beta_{(0)}$. Now adding $-Mw$ on both sides, we get

$$f(\xi, \beta_{(1)}, \beta^{'}_{(1)}) - f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) \geq -Mw - N \mid w^{'} \mid . \tag{4.33}$$

From equation (4.28) for $n = 0$ we can write

$$-\beta^{''}_{(1)} + \sqrt[3]{l} K(\xi) \beta^{'}_{(1)} + l\beta_{(1)} = -f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) + \sqrt[3]{l} K(\xi) \beta^{'}_{(0)} + l\beta_{(0)}.$$

Rearranging the above equation and adding $f(\xi, \beta_{(1)}, \beta^{'}_{(1)})$ on both sides, we have

$$\begin{aligned} f(\xi, \beta_{(1)}, \beta^{'}_{(1)}) - \beta^{''}_{(1)} =& f(\xi, \beta_{(1)}, \beta^{'}_{(1)}) - f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) \\ &+ \sqrt[3]{l} K(\xi)(\beta^{'}_{(0)} - \beta^{'}_{(1)}) + l(\beta_{(0)} - \beta_{(1)}). \end{aligned}$$

Using (4.33), we get

$$f(\xi, \beta_{(1)}, \beta'_{(1)}) - \beta''_{(1)} \geq -(M-l)w - (N\text{sign}(w') - \sqrt[3]{l}K(\xi))w'.$$

By Lemma3.2.13we know

$$(M-l)w + (N\text{sign}(w') - \sqrt[3]{l}K(\xi))w' \leq 0,$$

that is

$$-(M-l)w - (N\text{sign}(w') - \sqrt[3]{l}K(\xi))w' \geq 0.$$

So

$$f(\xi, \beta_{(1)}, \beta'_{(1)}) - \beta''_{(1)} \geq 0.$$

Therefore we conclude that $\beta_{(1)}$ is a supersolution.

Assume further that for some n, $\beta_{(n)}$ is a supersolution of (4.16), we will prove that $\beta_{(n+1)}$ is also an supersolution. Now rearranging and adding $\beta''_{(n+1)}$ on both sides of equation (4.28), we obtain

$$\begin{aligned}(\beta''_{(n)} - \beta''_{(n+1)}) - \sqrt[3]{l}K(\xi)(\beta'_{(n)} - \beta'_{(n+1)}) - l(\beta_{(n)} - \beta_{(n+1)}) \\ = -f(\xi, \beta_{(n)}, \beta'_{(n)}) + \beta''_{(n)}.\end{aligned}$$

Let $w = (\beta_{(n)} - \beta_{(n+1)})$ and we have

$$w'' - \sqrt[3]{l}K(\xi)w' - lw = -f(\xi, \beta_{(n+1)}, \beta'_{(n+1)}) + \beta''_{(n+1)}.$$

As $\beta_{(n)}$ is a supersolution we know $-f(\xi, \beta_{(n+1)}, \beta'_{(n+1)}) + \beta''_{(n+1)} \leq 0$ and hence

$$w'' - \sqrt[3]{l}K(\xi)w' - lw \leq 0.$$

By the construction of iterations $w \geq 0$ on $\partial\Omega_2$, so by Lemma3.2.4 $w \geq 0$ on Ω_2, that is, $\beta_{(n+1)} \geq \beta_{(n+2)}$ on Ω_2.

Now choosing $u = \beta_{(n+1)}$, $v_2 = \beta^{'}_{(n+1)}$ and $v_1 = \beta^{'}_{(n)}$, we have from (4.32)

$$-N \mid \beta^{'}_{(n+1)} - \beta^{'}_{(n)} \mid \leq f(\xi, \beta_{(n+1)}, \beta^{'}_{(n+1)}) - f(\xi, \beta_{(n+1)}, \beta^{'}_{(n)}).$$

Adding and subtracting $f(\xi, \beta_{(n)}, \beta^{'}_{(n)})$ on the left side we have

$$\begin{aligned} &f(\xi, \beta_{(n+1)}, \beta^{'}_{(n+1)}) - f(\xi, \beta_{(n)}, \beta^{'}_{(n)}) + \\ &f(\xi, \beta_{(n)}, \beta^{'}_{(n)}) - f(\xi, \beta_{(n+1)}, \beta^{'}_{(n)}) \geq -N \mid w^{'} \mid, \end{aligned}$$

where $w = \beta_{(n+1)} - \beta_{(n)}$. With the help of **[C2]** obtain the inequality

$$f(\xi, \beta_{(n+1)}, \beta^{'}_{(n+1)}) - f(\xi, \beta_{(n)}, \beta^{'}_{(n)}) + M(\beta_{(n)} - \beta_{(n+1)}) \geq -N \mid w^{'} \mid .$$

Adding $-Mw$ on both sides we get

$$f(\xi, \beta_{(n+1)}, \beta^{'}_{(n+1)}) - f(\xi, \beta_{(n)}, \beta^{'}_{(n)}) \geq -Mw - N \mid w^{'} \mid . \tag{4.34}$$

From equation (4.27) at iteration n we have

$$-\beta^{''}_{(n+1)} + \sqrt[3]{l}K(\xi)\beta^{'}_{(n+1)} + l\beta_{(n+1)} = -f(\xi, \beta_{(n)}, \beta^{'}_{(n)}) + \sqrt[3]{l}K(\xi)\beta^{'}_{(n)} + l\beta_{(n)}.$$

Rearranging and adding $f(\xi, \beta_{(n+1)}, \beta^{'}_{(n+1)})$ on both sides we get

$$\begin{aligned} -\beta^{''}_{(n+1)} + f(\xi, \beta_{(n+1)}, \beta^{'}_{(n+1)}) =& f(\xi, \beta_{(n+1)}, \beta^{'}_{(n+1)}) - f(\xi, \beta_{(n)}, \beta^{'}_{(n)}) \\ &+ \sqrt[3]{l}K(\xi)(\beta^{'}_{(n)} - \beta^{'}_{(n+1)}) + l(\beta_{(n)} - \beta_{(n+1)}). \end{aligned}$$

With the help of (4.34) we find

$$-\beta^{''}_{(n+1)} + f(\xi, \beta_{(n+1)}, \beta^{'}_{(n+1)}) \geq -(M - l)w - (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'},$$

where $w = \beta_{(n+1)} - \beta_{(n)}$. By Lemma3.2.13we know

$$(M - l)w + (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \leq 0,$$

that is

$$-(M - l)w - (N\text{sign}(w^{'}) - \sqrt[3]{l}K(\xi))w^{'} \geq 0.$$

Which implies

$$-\beta^{''}_{(n+2)} + f(\xi, \beta_{(n+2)}, \beta^{'}_{(n+2)}) \geq 0.$$

Hence for all n, β_{n+1} is a supersolution of (4.16). From the above analysis we can conclude that the functions $\beta_{(n+\frac{1}{2})}$ and $\beta_{(n+1)}$ defined by the iterations (4.27) and (4.27) are supersolutions of equation (4.16) and they are monotonically decreasing. ■

Lemma 4.2.3 *Assume $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n+1)}$ are the sequences defined by (4.17) and (4.18) respectively. Then for all $n \in \mathcal{N}$*

$$\underline{\alpha} \leq \alpha_{(n)} \leq \alpha_{(n+\frac{1}{2})} \leq \alpha_{(n+1)} \leq \alpha_{(n+\frac{3}{2})} \leq \bar{\beta} \quad \text{on } \Omega. \tag{4.35}$$

Here $\underline{\alpha}$ is a subsolution and $\bar{\beta}$ is a supersolution of (4.16).

Proof: We already know from Lemma4.2.1that $\alpha_{(n+\frac{1}{2})} \leq \alpha_{(n+\frac{3}{2})}$ and $\alpha_{(n)} \leq \alpha_{(n+1)}$. First we prove that for all $n \in \mathcal{N}$, $\alpha_{(n+\frac{1}{2})} \leq \alpha_{(n+1)} \leq \alpha_{(n+\frac{3}{2})}$. We proceed by induction. On $\Omega_1 \backslash \Omega_2$ $\alpha_{(\frac{1}{2})} = \alpha_{(1)}$ by definition. On $\Omega_{12} = \Omega_1 \cap \Omega_2$ subtract the defining equations for $\alpha_{(\frac{1}{2})}$ and $\alpha_{(1)}$ we obtain

$$\begin{aligned} -\alpha^{''}_{(1)} + \sqrt[3]{l}K(\xi)\alpha^{'}_{(1)} + l\alpha_{(1)} + \alpha^{''}_{(\frac{1}{2})} - \sqrt[3]{l}K(\xi)\alpha^{'}_{(\frac{1}{2})} - l\alpha_{(\frac{1}{2})} &= -f(\xi, \alpha_{(0)}, \alpha^{'}_{(0)}) \\ + \sqrt[3]{l}K(\xi)\alpha^{'}_{(0)} + l\alpha_{(0)} + f(\xi, \alpha_{(-\frac{1}{2})}, \alpha^{'}_{(-\frac{1}{2})}) &- \sqrt[3]{l}K(\xi)\alpha^{'}_{(-\frac{1}{2})} - l\alpha_{(-\frac{1}{2})}. \end{aligned}$$

Let $w = \alpha_{(1)} - \alpha_{(\frac{1}{2})}$, and as $\alpha_{(-\frac{1}{2})} = \alpha_{(0)}$ the above equation implies

$$-w^{''} + \sqrt[3]{l}K(\xi)w^{'} + lw = 0,$$

or

$$w^{''} - \sqrt[3]{l}K(\xi)w^{'} - lw = 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $\alpha_{(1)} \geq \alpha_{(\frac{1}{2})} = \alpha_{(0)}$. Furthermore on $\partial\Omega_2 \cap \Omega_1$, $\alpha_{(1)} = \alpha_{(\frac{1}{2})}$ by definition. Then we have $\alpha_{(1)} - \alpha_{(\frac{1}{2})} \geq 0$ on $\partial\Omega_{12}$. Hence we conclude by Lemma3.2.4that $\alpha_{(1)} \geq \alpha_{(\frac{1}{2})}$ on Ω_{12}. Since $\alpha_{(\frac{1}{2})} = \alpha_{(0)} \leq \alpha_{(1)}$ on $\Omega_2 \setminus \Omega_1$ then $\alpha_{(1)} \geq \alpha_{(\frac{1}{2})}$ on Ω as $\alpha_{(1)} = \alpha_{(\frac{1}{2})}$ on $\Omega_2 \backslash \Omega_1$ by definition.

On Ω_{12} the defining equations for $\alpha_{(\frac{3}{2})}$ and $\alpha_{(1)}$ are

$$-\alpha^{''}_{(\frac{3}{2})} + \sqrt[3]{l}K(\xi)\alpha^{'}_{(\frac{3}{2})} + l\alpha_{(\frac{3}{2})} = -f(\xi, \alpha_{(\frac{1}{2})}, \alpha^{'}_{(\frac{1}{2})}) + \sqrt[3]{l}K(\xi)\alpha^{'}_{(\frac{1}{2})} + l\alpha_{(\frac{1}{2})}, \tag{4.36}$$

and

$$-\alpha^{''}_{(1)} + \sqrt[3]{l}K(\xi)\alpha^{'}_{(1)} + l\alpha_{(1)} = -f(\xi, \alpha_{(0)}, \alpha^{'}_{(0)}) + \sqrt[3]{l}K(\xi)\alpha^{'}_{(0)} + l\alpha_{(0)}. \tag{4.37}$$

Now subtracting (4.37) from (4.36) and rearranging we have

$$\begin{aligned}&-(\alpha^{''}_{(\frac{3}{2})} - \alpha^{''}_{(1)}) + \sqrt[3]{l}K(\xi)(\alpha^{'}_{(\frac{3}{2})} - \alpha^{'}_{(1)}) + l(\alpha_{(\frac{3}{2})} - \alpha_{(1)}) \\ &= -f(\xi, \alpha_{(\frac{1}{2})}, \alpha^{'}_{(\frac{1}{2})}) + f(\xi, \alpha_{(0)}, \alpha^{'}_{(0)}) + \sqrt[3]{l}K(\xi)(\alpha^{'}_{(\frac{1}{2})} - \alpha^{'}_{(0)}) + l(\alpha_{(\frac{1}{2})} - \alpha_{(0)}).\end{aligned}$$

Let $w_1 = (\alpha_{(\frac{3}{2})} - \alpha_{(1)})$ and $w_2 = (\alpha_{(\frac{1}{2})} - \alpha_{(0)})$ then we obtain

$$-w^{''}_1 + \sqrt[3]{l}K(\xi)w^{'}_1 + lw_1 = -f(\xi, \alpha_{(\frac{1}{2})}, \alpha^{'}_{(\frac{1}{2})}) + f(\xi, \alpha_{(0)}, \alpha^{'}_{(0)}) + \sqrt[3]{l}K(\xi)w^{'}_2 + lw_2.$$

Adding and subtracting $f(\xi, \alpha_{(0)}, \alpha^{'}_{(\frac{1}{2})})$ on the left side of the above equation we get

$$\begin{aligned}-w^{''}_1 + \sqrt[3]{l}K(\xi)w^{'}_1 + lw_1 &= -f(\xi, \alpha_{(\frac{1}{2})}, \alpha^{'}_{(\frac{1}{2})}) + f(\xi, \alpha_{(0)}, \alpha^{'}_{(\frac{1}{2})}) \\ &\quad - f(\xi, \alpha_{(0)}, \alpha^{'}_{(\frac{1}{2})}) + f(\xi, \alpha_{(0)}, \alpha^{'}_{(0)}) + \sqrt[3]{l}K(\xi)w^{'}_2 + lw_2.\end{aligned}$$

Since $\alpha_{(\frac{1}{2})} \geq \alpha_{(0)}$ on Ω_{12} then by **[C2]** and **[C3]** we can write

$$-w_1^{''} + \sqrt[3]{l}K(\xi)w_1^{'} + lw_1 \geq -M(\alpha_{(\frac{1}{2})} - \alpha_{(0)}) - N \mid \alpha_{(\frac{1}{2})}^{'} - \alpha_{(0)}^{'} \mid + \sqrt[3]{l}K(\xi)w_2^{'} + lw_2,$$

or

$$-w_1^{''} + \sqrt[3]{l}K(\xi)w_1^{'} + lw_1 \geq -Mw_2 - N \mid w_2^{'} \mid + \sqrt[3]{l}K(\xi)w_2^{'} + lw_2.$$

Multiplying both sides by -1 and using the fact that $\mid w_2^{'} \mid = \text{sign}(w_2^{'})w_2^{'}$, we have

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq (M-l)w_2 + (N \text{ sign } (w_2^{'}) - \sqrt[3]{l}K(\xi))w_2^{'}. \tag{4.38}$$

By Lemma3.2.13we conclude from equation (4.38) that

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $\alpha_{(\frac{3}{2})} \geq \alpha_{(1)} = \alpha_{(\frac{1}{2})}$ and on $\partial\Omega_2 \cap \Omega_1$ $\alpha_{(1)} = \alpha_{(\frac{1}{2})}$ by definition. This implies that $w_1 = \alpha_{(\frac{3}{2})} - \alpha_{(1)} \geq 0$ on $\partial\Omega_{12}$. Hence by Lemma3.2.4we can say that $\alpha_{(\frac{3}{2})} \geq \alpha_{(1)}$ on Ω_{12}. Since $\alpha_{(1)} = \alpha_{(\frac{1}{2})} \leq \alpha_{(\frac{3}{2})}$ on $\Omega_1 \setminus \Omega_2$, we conclude that $\alpha_{(\frac{3}{2})} \geq \alpha_{(1)}$ on Ω. That implies (4.35) is true for $n = 1$.

Assume (4.35) is true for some n, now we will prove (4.35) holds for $n+1$. On $\Omega_1 \backslash \Omega_2$, $\alpha_{(n+\frac{1}{2})} = \alpha_{(n+1)}$ by definition. On Ω_{12} subtract the defining equations for $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n+1)}$ we obtain

$$\begin{aligned}
&-\alpha_{(n+1)}^{''} + \sqrt[3]{l}K(\xi)\alpha_{(n+1)}^{'} + l\alpha_{(n+1)} + \alpha_{(n+\frac{1}{2})}^{''} - \sqrt[3]{l}K(\xi)\alpha_{(n+\frac{1}{2})}^{'} - l\alpha_{(n+\frac{1}{2})} \\
&= -f(\xi, \alpha_{(n)}, \alpha_{(n)}^{'}) + \sqrt[3]{l}K(\xi)\alpha_{(n)}^{'} + l\alpha_{(n)} + f(\xi, \alpha_{(n-\frac{1}{2})}, \alpha_{(n-\frac{1}{2})}^{'}) - \sqrt[3]{l}K(\xi)\alpha_{(n-\frac{1}{2})}^{'} \\
&- l\alpha_{(n-\frac{1}{2})}.
\end{aligned}$$

Rearranging we have

$$\begin{aligned}
&-(\alpha_{(n+1)}^{''} - \alpha_{(n+\frac{1}{2})}^{''}) + \sqrt[3]{l}K(\xi)(\alpha_{(n+1)}^{'} - \alpha_{(n+\frac{1}{2})}^{'}) + l(\alpha_{(n+1)} - \alpha_{(n+\frac{1}{2})}) \\
&= -f(\xi, \alpha_{(n)}, \alpha_{(n)}^{'}) + f(\xi, \alpha_{(n-\frac{1}{2})}, \alpha_{(n-\frac{1}{2})}^{'}) + \sqrt[3]{l}K(\xi)(\alpha_{(n)}^{'} - \alpha_{(n-\frac{1}{2})}^{'}) + l(\alpha_{(n)} - \alpha_{(n-\frac{1}{2})}).
\end{aligned}$$

Let $w_1 = \alpha_{(n+1)} - \alpha_{(n+\frac{1}{2})}$ and $w_2 = \alpha_{(n)} - \alpha_{(n-\frac{1}{2})}$ we have

$$-w_1'' + \sqrt[3]{l}K(\xi)w_1' + lw_1 = -f(\xi, \alpha_{(n)}, \alpha_{(n)}') + f(\xi, \alpha_{(n-\frac{1}{2})}, \alpha_{(n-\frac{1}{2})}') + \sqrt[3]{l}K(\xi)w_2' + lw_2.$$

Adding and subtracting $f(\xi, \alpha_{(n-\frac{1}{2})}, \alpha_{(n)}')$ on the left side of the above equation we obtain

$$\begin{aligned} -w_1'' + \sqrt[3]{l}K(\xi)w_1' + lw_1 = &- f(\xi, \alpha_{(n)}, \alpha_{(n)}') + f(\xi, \alpha_{(n-\frac{1}{2})}, \alpha_{(n)}') - f(\xi, \alpha_{(n-\frac{1}{2})}, \alpha_{(n)}') \\ &+ f(\xi, \alpha_{(n-\frac{1}{2})}, \alpha_{(n-\frac{1}{2})}') + \sqrt[3]{l}K(\xi)w_2' + lw_2. \end{aligned}$$

Now by **[C2]** and **[C3]** we have

$$-w_1'' + \sqrt[3]{l}K(\xi)w_1' + lw_1 \geq -M(\alpha_{(n)} - \alpha_{(n-\frac{1}{2})}) - N \mid \alpha_{(n)}' - \alpha_{(n-\frac{1}{2})}' \mid + \sqrt[3]{l}K(\xi)w_2' + lw_2.$$

Multiplying both sides by -1 we obtain

$$w_1'' - \sqrt[3]{l}K(\xi)w_1' - lw_1 \leq (M - l)w_2 + (N\text{sign}(w_2') - \sqrt[3]{l}K(\xi))w_2'.$$

By Lemma3.2.13we conclude that

$$w_1'' - \sqrt[3]{l}K(\xi)w_1' - lw_1 \leq 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $\alpha_{(n+1)} \geq \alpha_{(n)} = \alpha_{(n+\frac{1}{2})}$ while on $\partial\Omega_2 \cap \Omega_1$ $\alpha_{(n+1)} = \alpha_{(n+\frac{1}{2})}$ by definition. This implies $\alpha_{(n+1)} - \alpha_{(n+\frac{1}{2})} \geq 0$ on $\partial\Omega_{12}$. Hence by Lemma3.2.4we conclude that $\alpha_{(n+1)} \geq \alpha_{(n+\frac{1}{2})}$ on Ω_{12}. Since $\alpha_{(n+\frac{1}{2})} = \alpha_{(n)} \leq \alpha_{(n+1)}$ on $\Omega_2 \setminus \Omega_1$, $\alpha_{(n+1)} \geq \alpha_{(n+\frac{1}{2})}$ on Ω. Proceeding in this way for $\alpha_{(n+\frac{3}{2})}$ and $\alpha_{(n+1)}$ we conclude that $\alpha_{(n+1)} \leq \alpha_{(n+\frac{3}{2})}$.

In Lemma4.2.1we have proved that $\alpha_{(n+\frac{1}{2})}$ and $\alpha_{(n)}$ are monotonically increasing from subsolution $\underline{\alpha}$ in Ω, so these sequences are bounded below by $\underline{\alpha}$ in Ω. Hence we only need to show that for all $n \in \mathcal{N}$, $\alpha_{(n+\frac{1}{2})} \leq \bar{\beta}$. From (4.17) for $n = 0$, we have

$$\begin{aligned} -\alpha_{(\frac{1}{2})}'' + \sqrt[3]{l}K(\xi)\alpha_{(\frac{1}{2})}' + l\alpha_{(\frac{1}{2})} = &- f(\xi, \alpha_{(0)}, \alpha_{(0)}') \\ &+ \sqrt[3]{l}K(\xi)\alpha_{(0)}' + l\alpha_{(0)}, \end{aligned} \tag{4.39}$$

as $\alpha_{(-\frac{1}{2})} = \alpha_{(0)} = \underline{\alpha}$. Also we know $\bar{\beta}$ is a supersolution, so

$$\bar{\beta}'' \leq f(\xi, \bar{\beta}, \bar{\beta}').$$

Adding $-\sqrt[3]{l}K(\xi)\bar{\beta}' - l\bar{\beta}$ on both sides of the above inequality, we get

$$\bar{\beta}'' - \sqrt[3]{l}K(\xi)\bar{\beta}' - l\bar{\beta} \leq f(\xi, \bar{\beta}, \bar{\beta}') - \sqrt[3]{l}K(\xi)\bar{\beta}' - l\bar{\beta}.$$

Multiplying both sides by -1, we have

$$-\bar{\beta}'' + \sqrt[3]{l}K(\xi)\bar{\beta}' + l\bar{\beta} \geq -f(\xi, \bar{\beta}, \bar{\beta}') + \sqrt[3]{l}K(\xi)\bar{\beta}' + l\bar{\beta}. \tag{4.40}$$

Now subtracting (4.39) from (4.40) and rearranging, we find

$$\begin{aligned} -(\bar{\beta}'' - \alpha''_{(\frac{1}{2})}) + \sqrt[3]{l}K(\xi)(\bar{\beta}' - \alpha'_{(\frac{1}{2})}) + l(\bar{\beta} - \alpha_{(\frac{1}{2})}) \geq -f(\xi, \bar{\beta}, \bar{\beta}') + \\ f(\xi, \alpha_{(0)}, \alpha'_{(0)}) + \sqrt[3]{l}K(\xi)(\bar{\beta}' - \alpha'_{(0)}) + l(\bar{\beta} - \alpha_{(0)}). \end{aligned}$$

Adding and subtracting $f(\xi, \bar{\beta}, \alpha'_{(0)})$ on the right hand side we get

$$\begin{aligned} &-(\bar{\beta}'' - \alpha''_{(\frac{1}{2})}) + \sqrt[3]{l}K(\xi)(\bar{\beta}' - \alpha'_{(\frac{1}{2})}) + l(\bar{\beta} - \alpha_{(\frac{1}{2})}) \geq -f(\xi, \bar{\beta}, \bar{\beta}') + \\ &f(\xi, \bar{\beta}, \alpha'_{(0)}) - f(\xi, \bar{\beta}, \alpha'_{(0)}) + f(\xi, \alpha_{(0)}, \alpha'_{(0)}) + \sqrt[3]{l}K(\xi)(\bar{\beta}' - \alpha'_{(0)}) + l(\bar{\beta} - \alpha_{(0)}). \end{aligned}$$

Since $\bar{\beta} \geq \alpha_{(0)} = \underline{\alpha}$ then by **[C2]** and **[C3]** we can write

$$-w_1'' + \sqrt[3]{l}K(\xi)w_1' + lw_1 \geq -N \mid \bar{\beta}' - \alpha'_{(0)} \mid -M(\bar{\beta} - \alpha_{(0)}) + \sqrt[3]{l}K(\xi)w_2' + lw_2,$$

or

$$-w_1'' + \sqrt[3]{l}K(\xi)w_1' + lw_1 \geq -N \mid w_2' \mid -Mw_2 + \sqrt[3]{l}K(\xi)w_2' + lw_2,$$

where $w_1 = \bar{\beta} - \alpha_{(\frac{1}{2})}$ and $w_2 = \bar{\beta} - \alpha_{(0)}$. Multiplying both sides by -1 and using the fact that $\mid w_2' \mid = \text{sign}(w_2')w_2'$ we obtain

$$w_1'' - \sqrt[3]{l}K(\xi)w_1' - lw_1 \leq (M - l)w_2 + (N\text{sign}(w_2') - \sqrt[3]{l}K(\xi))w_2'. \tag{4.41}$$

By Lemma3.2.13we conclude from (4.41) that

$$w_1'' - \sqrt[3]{l}K(\xi)w_1' - lw_1 \leq 0.$$

As $\bar{\beta} \geq \alpha_{(\frac{1}{2})}$ on $\partial\Omega$ by Lemma3.2.4we conclude that $\bar{\beta} \geq \alpha_{(\frac{1}{2})}$ on Ω. That is $\bar{\beta} \geq \alpha_{(\frac{1}{2})}$ on Ω and hence the base induction step is established.

Assume that for some n, $\bar{\beta} \geq \alpha_{(n+\frac{1}{2})}$ we will prove it is true for $n+1$, that is $\bar{\beta} \geq \alpha_{(n+\frac{3}{2})}$. Now using the definition of equation for $\alpha_{(n+\frac{3}{2})}$ and subtracting it from (4.40) we get

$$\begin{aligned}-(\bar{\beta}'' - \alpha''_{(n+\frac{3}{2})}) + \sqrt[3]{l}K(\xi)(\bar{\beta}' - \alpha'_{(n+\frac{3}{2})}) + l(\bar{\beta} - \alpha_{(n+\frac{3}{2})}) = -f(\xi, \bar{\beta}, \bar{\beta}') + \\ f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) + \sqrt[3]{l}K(\xi)(\bar{\beta}' - \alpha'_{(n+\frac{1}{2})}) + l(\bar{\beta} - \alpha_{(n+\frac{1}{2})}).\end{aligned}$$

Adding and subtracting $f(\xi, \bar{\beta}, \alpha'_{(n+\frac{1}{2})})$ on the right hand side we have

$$\begin{aligned}-(\bar{\beta}'' - \alpha''_{(n+\frac{3}{2})}) + \sqrt[3]{l}K(\xi)(\bar{\beta}' - \alpha'_{(n+\frac{3}{2})}) + l(\bar{\beta} - \alpha_{(n+\frac{3}{2})}) &= -f(\xi, \bar{\beta}, \bar{\beta}') \\ &+ f(\xi, \bar{\beta}, \alpha'_{(n+\frac{1}{2})}) - f(\xi, \bar{\beta}, \alpha'_{(n+\frac{1}{2})}) + f(\xi, \alpha_{(n+\frac{1}{2})}, \alpha'_{(n+\frac{1}{2})}) \\ &+ \sqrt[3]{l}K(\xi)(\bar{\beta}' - \alpha'_{(n+\frac{1}{2})}) + l(\bar{\beta} - \alpha_{(n+\frac{1}{2})}).\end{aligned}$$

Since $\bar{\beta} \geq \alpha_{(n+\frac{1}{2})}$ then by **[C2]** and **[C3]** we can write

$$-w_1'' + \sqrt[3]{l}K(\xi)w_1' + lw_1 \geq -N \mid \bar{\beta}' - \alpha'_{(n+\frac{1}{2})} \mid -M(\bar{\beta} - \alpha_{(n+\frac{1}{2})}) + \sqrt[3]{l}K(\xi)w_2' + lw_2,$$

or

$$-w_1'' + \sqrt[3]{l}K(\xi)w_1' + lw_1 \geq -N \mid w_2' \mid -Mw_2 + \sqrt[3]{l}K(\xi)w_2' + lw_2,$$

where $w_1 = \bar{\beta} - \alpha_{(n+\frac{3}{2})}$ and $w_2 = \bar{\beta} - \alpha_{(n+\frac{1}{2})}$. Multiplying both sides by -1 and using the fact that $\mid w_2' \mid = \text{sign}(w_2')w_2'$, we obtain

$$w_1'' - \sqrt[3]{l}K(\xi)w_1' - lw_1 \leq (M-l)w_2 + (N\text{sign}(w_2') - \sqrt[3]{l}K(\xi))w_2'. \tag{4.42}$$

By Lemma3.2.13we conclude from (4.42) that

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq 0.$$

As $\bar{\beta} \geq \alpha_{(n+\frac{3}{2})}$ on $\partial\Omega$, by Lemma3.2.4we conclude that $\bar{\beta} \geq \alpha_{(n+\frac{3}{2})}$ on Ω. Hence $\bar{\beta} \geq \alpha_{(n+\frac{1}{2})}$ for all $n \in \mathcal{N}$. This completes the proof of (4.35). ■

Lemma 4.2.4 *Assume $\beta_{(n+\frac{1}{2})}$ and $\beta_{(n+1)}$ are the sequences defined by* (4.27) *and* (4.28) *respectively. Then for all $n \in \mathcal{N}$*

$$\bar{\beta} \geq \beta_{(n)} \geq \beta_{(n+\frac{1}{2})} \geq \beta_{(n+1)} \geq \beta_{(n+\frac{3}{2})} \geq \underline{\alpha} \quad \text{on } \Omega. \tag{4.43}$$

Here $\underline{\alpha}$ is a subsolution and $\bar{\beta}$ is a supersolution of (4.16).

Proof: We already know from Lemma4.2.2that $\beta_{(n+\frac{1}{2})} \geq \beta_{(n+\frac{3}{2})}$ and $\beta_{(n)} \geq \beta_{(n+1)}$. First we show that for all $n \in \mathcal{N}$, $\beta_{(n+\frac{1}{2})} \geq \beta_{(n+1)} \geq \beta_{(n+\frac{3}{2})}$. We proceed by induction. On $\Omega_1 \backslash \Omega_2$, $\beta_{(\frac{1}{2})} = \beta_{(1)}$ by definition. On $\Omega_{12} = \Omega_1 \cap \Omega_2$ subtract the defining equations for $\beta_{(\frac{1}{2})}$ and $\beta_{(1)}$ we obtain

$$\begin{aligned} -\beta_{(\frac{1}{2})}^{''} + \sqrt[3]{l}K(\xi)\beta_{(\frac{1}{2})}^{'} + l\beta_{(\frac{1}{2})} + \beta_{(1)}^{''} - \sqrt[3]{l}K(\xi)\beta_{(1)}^{'} - l\beta_{(1)} &= -f(\xi, \beta_{(-\frac{1}{2})}, \beta_{(-\frac{1}{2})}^{'}) \\ + \sqrt[3]{l}K(\xi)\beta_{(-\frac{1}{2})}^{'} + l\beta_{(-\frac{1}{2})} + f(\xi, \beta_{(0)}, \beta_{(0)}^{'}) &- \sqrt[3]{l}K(\xi)\beta_{(0)}^{'} - l\beta_{(0)}. \end{aligned}$$

Let $w = \beta_{(\frac{1}{2})} - \beta_{(1)}$, as $\beta_{(-\frac{1}{2})} = \beta_{(0)}$ the above equation implies

$$-w^{''} + \sqrt[3]{l}K(\xi)w^{'} + lw = 0,$$

or

$$w^{''} - \sqrt[3]{l}K(\xi)w^{'} - lw = 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $\beta_{(\frac{1}{2})} = \beta_{(0)} \geq \beta_{(1)}$. Furthermore on $\partial\Omega_2 \cap \Omega_1$, $\beta_{(1)} = \beta_{(\frac{1}{2})}$ by definition. This implies that $\beta_{(\frac{1}{2})} - \beta_{(1)} \geq 0$ on $\partial\Omega_{12}$. Hence we conclude by Lemma3.2.4that

$\beta_{(\frac{1}{2})} \geq \beta_{(1)}$ on Ω_{12}. Now by the definition of iteration, on $\Omega_2 \setminus \Omega_1$, $\beta_{(\frac{1}{2})}$ is defined as $\beta_{(0)}$, so $\beta_{(\frac{1}{2})} = \beta_{(0)} \geq \beta_{(1)}$ on $\Omega_2 \setminus \Omega_1$. Also by the definition of iteration, $\beta_{(1)}$ is defined as $\beta_{(\frac{1}{2})}$ on $\Omega_1 \setminus \Omega_2$, so $\beta_{(1)} = \beta_{(\frac{1}{2})}$ on $\Omega_1 \setminus \Omega_2$. Hence we conclude $\beta_{(\frac{1}{2})} \geq \beta_{(1)}$ on Ω.

On $\Omega_{12} = \Omega_1 \cap \Omega_2$ the defining equations for $\beta_{(1)}$ and $\beta_{(\frac{3}{2})}$ are

$$-\beta''_{(1)} + \sqrt[3]{l}K(\xi)\beta'_{(1)} + l\beta_{(1)} = -f(\xi, \beta_{(0)}, \beta'_{(0)}) + \sqrt[3]{l}K(\xi)\beta'_{(0)} + l\beta_{(0)}, \tag{4.44}$$

and

$$-\beta''_{(\frac{3}{2})} + \sqrt[3]{l}K(\xi)\beta'_{(\frac{3}{2})} + l\beta_{(\frac{3}{2})} = -f(\xi, \beta_{(\frac{1}{2})}, \beta'_{(\frac{1}{2})}) + \sqrt[3]{l}K(\xi)\beta'_{(\frac{1}{2})} + l\beta_{(\frac{1}{2})}. \tag{4.45}$$

Now subtracting (4.45) from (4.44) and rearranging we have

$$\begin{aligned}
&-(\beta''_{(1)} - \beta''_{(\frac{3}{2})}) + \sqrt[3]{l}K(\xi)(\beta'_{(1)} - \beta'_{(\frac{3}{2})}) + l(\beta_{(1)} - \beta_{(\frac{3}{2})}) \\
&= -f(\xi, \beta_{(0)}, \beta'_{(0)}) + f(\xi, \beta_{(\frac{1}{2})}, \beta'_{(\frac{1}{2})}) + \sqrt[3]{l}K(\xi)(\beta'_{(0)} - \beta'_{(\frac{1}{2})}) + l(\beta_{(0)} - \beta_{(\frac{1}{2})}).
\end{aligned}$$

Let $w_1 = \beta_{(1)} - \beta_{(\frac{3}{2})}$ and $w_2 = \beta_{(0)} - \beta_{(\frac{1}{2})}$ then we obtain

$$-w''_1 + \sqrt[3]{l}K(\xi)w'_1 + lw_1 = -f(\xi, \beta_{(0)}, \beta'_{(0)}) + f(\xi, \beta_{(\frac{1}{2})}, \beta'_{(\frac{1}{2})}) + \sqrt[3]{l}K(\xi)w'_2 + lw_2.$$

Adding and subtracting $f(\xi, \beta_{(\frac{1}{2})}, \beta'_{(0)})$ on the left side of the above equation we get

$$\begin{aligned}
-w''_1 + \sqrt[3]{l}K(\xi)w'_1 + lw_1 &= -f(\xi, \beta_{(0)}, \beta'_{(0)}) + f(\xi, \beta_{(\frac{1}{2})}, \beta'_{(0)}) \\
&\quad - f(\xi, \beta_{(\frac{1}{2})}, \beta'_{(0)}) + f(\xi, \beta_{(\frac{1}{2})}, \beta'_{(\frac{1}{2})}) + \sqrt[3]{l}K(\xi)w'_2 + lw_2.
\end{aligned}$$

Since $\beta_{(\frac{1}{2})} \leq \beta_{(0)}$ on Ω_{12} then by **[C2]** and **[C3]** we can write

$$-w''_1 + \sqrt[3]{l}K(\xi)w'_1 + lw_1 \geq -M(\beta_{(0)} - \beta_{(\frac{1}{2})}) - N \mid \beta'_{(0)} - \beta'_{(\frac{1}{2})} \mid + \sqrt[3]{l}K(\xi)w'_2 + lw_2,$$

or

$$-w''_1 + \sqrt[3]{l}K(\xi)w'_1 + lw_1 \geq -Mw_2 - N \mid w'_2 \mid + \sqrt[3]{l}K(\xi)w'_2 + lw_2.$$

Multiplying both sides by -1 and using the fact that $| w_2^{'} |= \text{sign}(w_2^{'})w_2^{'}$, we have

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq (M-l)w_2 + (N \text{ sign } (w_2^{'}) - \sqrt[3]{l}K(\xi))w_2^{'}. \tag{4.46}$$

By Lemma3.2.13we conclude from equation (4.46) that

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq 0.$$

On $\partial\Omega_1 \cap \Omega_2$, $\beta_{(\frac{3}{2})} \leq \beta_{(1)} = \beta_{(\frac{1}{2})}$ and on $\partial\Omega_2 \cap \Omega_1$, $\beta_{(1)} = \beta_{(\frac{1}{2})}$ by definition. This implies that $w_1 = \beta_{(1)} - \beta_{(\frac{3}{2})} \geq 0$ on $\partial\Omega_{12}$. Hence by Lemma3.2.4we can say that $\beta_{(1)} \geq \beta_{(\frac{3}{2})}$ on Ω_{12}. Since $\beta_{(1)} = \beta_{(\frac{1}{2})} \geq \beta_{(\frac{3}{2})}$ on $\Omega_1 \setminus \Omega_2$ therefore we conclude that $\beta_{(1)} \geq \beta_{(\frac{3}{2})}$ on Ω. That implies (4.43) is true for $n = 1$.

Assume (4.43) is true for some n, now we will prove (4.43) holds for $n+1$. On $\Omega_1 \backslash \Omega_2$, $\beta_{(n+\frac{1}{2})} = \beta_{(n+1)}$ by definition. On $\Omega_{12} = \Omega_1 \cap \Omega_2$ subtract the defining equations for $\beta_{(n+\frac{1}{2})}$ and $\beta_{(n+1)}$ to obtain

$$\begin{aligned}
&- \beta_{(n+\frac{1}{2})}^{''} + \sqrt[3]{l}K(\xi)\beta_{(n+\frac{1}{2})}^{'} + l\beta_{(n+\frac{1}{2})} + \beta_{(n+1)}^{''} - \sqrt[3]{l}K(\xi)\beta_{(n+1)}^{'} - l\beta_{(n+1)} \\
&= -f(\xi, \beta_{(n-\frac{1}{2})}, \beta_{(n-\frac{1}{2})}^{'}) + \sqrt[3]{l}K(\xi)\beta_{(n-\frac{1}{2})}^{'} + l\beta_{(n-\frac{1}{2})} \\
&+ f(\xi, \beta_{(n)}, \beta_{(n)}^{'}) - \sqrt[3]{l}K(\xi)\beta_{(n)}^{'} - l\beta_{(n)}.
\end{aligned}$$

Rearranging we have

$$\begin{aligned}
&- (\beta_{(n+\frac{1}{2})}^{''} - \beta_{(n+1)}^{''}) + \sqrt[3]{l}K(\xi)(\beta_{(n+\frac{1}{2})}^{'} - \beta_{(n+1)}^{'}) + l(\beta_{(n+\frac{1}{2})} - \beta_{(n+1)}) \\
&= -f(\xi, \beta_{(n-\frac{1}{2})}, \beta_{(n-\frac{1}{2})}^{'}) + f(\xi, \beta_{(n)}, \beta_{(n)}^{'}) + \sqrt[3]{l}K(\xi)(\beta_{(n-\frac{1}{2})}^{'} - \beta_{(n)}^{'}) + l(\beta_{(n-\frac{1}{2})} - \beta_{(n)}).
\end{aligned}$$

Let $w_1 = \beta_{(n+\frac{1}{2})}^{''} - \beta_{(n+1)}^{''}$ and $w_2 = \beta_{(n-\frac{1}{2})} - \beta_{(n)}$ we have

$$-w_1^{''} + \sqrt[3]{l}K(\xi)w_1^{'} + lw_1 = -f(\xi, \beta_{(n-\frac{1}{2})}, \beta_{(n-\frac{1}{2})}^{'}) + f(\xi, \beta_{(n)}, \beta_{(n)}^{'}) + \sqrt[3]{l}K(\xi)w_2^{'} + lw_2.$$

Adding and subtracting $f(\xi, \beta_{(n)}, \beta^{'}_{(n-\frac{1}{2})})$ on the left side of the above equation we obtain

$$
\begin{aligned}
-w_1^{''} + \sqrt[3]{l}K(\xi)w_1^{'} + lw_1 = &-f(\xi, \beta_{(n-\frac{1}{2})}, \beta^{'}_{(n-\frac{1}{2})}) + f(\xi, \beta_{(n)}, \beta^{'}_{(n-\frac{1}{2})}) \\
&- f(\xi, \beta_{(n)}, \beta^{'}_{(n-\frac{1}{2})}) + f(\xi, \beta_{(n)}, \beta^{'}_{(n)}) + \sqrt[3]{l}K(\xi)w_2^{'} + lw_2.
\end{aligned}
$$

Now by **[C2]** and **[C3]** we have

$$
\begin{aligned}
-w_1^{''} + \sqrt[3]{l}K(\xi)w_1^{'} + lw_1 \geq &- M(\beta_{(n-\frac{1}{2})} - \beta_{(n)}) - N \mid \beta^{'}_{(n-\frac{1}{2})} - \beta^{'}_{(n)} \mid \\
&+ \sqrt[3]{l}K(\xi)w_2^{'} + lw_2.
\end{aligned}
$$

Multiplying both sides by -1 and using the fact that $\mid w_2^{'} \mid = \text{sign}(w_2^{'})w_2^{'}$, we obtain

$$
w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq (M-l)w_2 + (N\text{sign}(w_2^{'}) - \sqrt[3]{l}K(\xi))w_2^{'}.
$$

By Lemma3.2.13we conclude that

$$
w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq 0.
$$

On $\partial\Omega_1 \cap \Omega_2$, $\beta_{(n+\frac{1}{2})} = \beta_{(n)} \geq \beta_{(n+1)}$ while on $\partial\Omega_2 \cap \Omega_1$, $\beta_{(n+1)} = \beta_{(n+\frac{1}{2})}$ by definition. This implies $\beta_{(n+\frac{1}{2})} - \beta_{(n+1)} \geq 0$ on $\partial\Omega_{12}$. Hence by Lemma3.2.4we conclude that $\beta_{(n+\frac{1}{2})} \geq \beta_{(n)}$ on Ω_{12}. Since $\beta_{(n+\frac{1}{2})} = \beta_{(n)} \geq \beta_{(n+1)}$ on $\Omega_2 \setminus \Omega_1$ then $\beta_{(n+\frac{1}{2})} \geq \beta_{(n+1)}$ on Ω. Repeating these steps for $\beta_{(n+\frac{3}{2})}$ and $\beta_{(n+1)}$ we conclude that $\beta_{(n+1)} \geq \beta_{(n+\frac{3}{2})}$.

In Lemma4.2.2we have proved that $\beta_{(n+\frac{1}{2})}$ and $\beta_{(n)}$ are monotonically decreasing from supersolution $\bar{\beta}$ in Ω, so these sequences are bounded above by $\bar{\beta}$ in Ω. Then we only need to show that for all $n \in \mathcal{N}$, $\beta_{(n+\frac{1}{2})} \geq \underline{\alpha}$. From (4.27) for $n = 0$, we have

$$
\begin{aligned}
-\beta^{''}_{(\frac{1}{2})} + \sqrt[3]{l}K(\xi)\beta^{'}_{(\frac{1}{2})} + l\beta_{(\frac{1}{2})} = &- f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) \\
&+ \sqrt[3]{l}K(\xi)\beta^{'}_{(0)} + l\beta_{(0)}, \qquad (4.47)
\end{aligned}
$$

as $\beta_{(-\frac{1}{2})} = \beta_{(0)} = \bar{\beta}$. Also we know $\underline{\alpha}$ is a subsolution, so

$$\underline{\alpha}^{''} \geq f(\xi,\underline{\alpha},\underline{\alpha}^{'}).$$

Adding $-\sqrt[3]{l}K(\xi)\underline{\alpha}^{'} - l\underline{\alpha}$ on both sides of the above inequality, we get

$$\underline{\alpha}^{''} - \sqrt[3]{l}K(\xi)\underline{\alpha}^{'} - l\underline{\alpha} \geq f(\xi,\underline{\alpha},\underline{\alpha}^{'}) - \sqrt[3]{l}K(\xi)\underline{\alpha}^{'} - l\underline{\alpha}. \tag{4.48}$$

Now adding (4.47) and (4.48), we have

$$-(\beta^{''}_{(\frac{1}{2})} - \underline{\alpha}^{''}) + \sqrt[3]{l}K(\xi)(\beta^{'}_{(\frac{1}{2})} - \underline{\alpha}^{'}) + l(\beta_{(\frac{1}{2})} - \underline{\alpha}) \geq -f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) +$$
$$f(\xi,\underline{\alpha},\underline{\alpha}^{'}) + \sqrt[3]{l}K(\xi)(\beta^{'}_{(0)} - \underline{\alpha}^{'}) + l(\beta_{(0)} - \underline{\alpha}).$$

Adding and subtracting $f(\xi, \beta_{(0)},\underline{\alpha})$ on right hand side we get

$$-(\beta^{''}_{(\frac{1}{2})} - \underline{\alpha}^{''}) + \sqrt[3]{l}K(\xi)(\beta^{'}_{(\frac{1}{2})} - \underline{\alpha}^{'}) + l(\beta_{(\frac{1}{2})} - \underline{\alpha}) \geq -f(\xi, \beta_{(0)}, \beta^{'}_{(0)}) +$$
$$f(\xi, \beta_{(0)},\underline{\alpha}) - f(\xi, \beta_{(0)},\underline{\alpha}) + f(\xi,\underline{\alpha},\underline{\alpha}^{'}) + \sqrt[3]{l}K(\xi)(\beta^{'}_{(0)} - \underline{\alpha}^{'}) + l(\beta_{(0)} - \underline{\alpha}).$$

Since $\bar{\beta} \geq \alpha_{(0)} = \underline{\alpha}$ then by **[C2]** and **[C3]** we can write

$$- w_1^{''} + \sqrt[3]{l}K(\xi)w_1^{'} + lw_1 \geq -N \mid \beta^{'}_{(0)} - \underline{\alpha}^{'} \mid -M(\beta_{(0)} - \underline{\alpha}) + \sqrt[3]{l}K(\xi)w_2^{'} + lw_2,$$

or

$$- w_1^{''} + \sqrt[3]{l}K(\xi)w_1^{'} + lw_1 \geq -N \mid w_2^{'} \mid -Mw_2 + \sqrt[3]{l}K(\xi)w_2^{'} + lw_2,$$

where $w_1 = \beta_{(\frac{1}{2})} - \underline{\alpha}$ and $w_2 = \beta_{(0)} - \underline{\alpha}$. Multiplying both sides by -1 and using the fact that $\mid w_2^{'} \mid = \mathrm{sign}(w_2^{'})w_2^{'}$ we obtain

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq (M - l)w_2 + (N\mathrm{sign}(w_2^{'}) - \sqrt[3]{l}K(\xi))w_2^{'}. \tag{4.49}$$

By Lemma3.2.13we conclude from (4.49) that

$$w_1^{''} - \sqrt[3]{l}K(\xi)w_1^{'} - lw_1 \leq 0.$$

As $\underline{\alpha} \leq \beta_{(\frac{1}{2})}$ on $\partial\Omega$, by Lemma3.2.4we conclude that $\underline{\alpha} \leq \beta_{(\frac{1}{2})}$ on Ω. That is $\underline{\alpha} \leq \beta_{(\frac{1}{2})}$ on Ω. Hence the base induction step is established.

Assume that for some n, $\underline{\alpha} \leq \beta_{(n+\frac{1}{2})}$ we will prove it is true for $n+1$, that is $\underline{\alpha} \leq \beta_{(n+\frac{3}{2})}$. Now using the definition of equation for $\beta_{(n+\frac{3}{2})}$ and adding to (4.48), we obtain

$$-(\beta''_{(n+\frac{3}{2})} - \underline{\alpha}'') + \sqrt[3]{l}K(\xi)(\beta'_{(n+\frac{3}{2})} - \underline{\alpha}') + l(\beta_{(n+\frac{3}{2})} - \underline{\alpha}) \geq -f(\xi, \beta_{(n+\frac{1}{2})}, \beta'_{(n+\frac{1}{2})}) + f(\xi, \underline{\alpha}, \underline{\alpha}') + \sqrt[3]{l}K(\xi)(\beta'_{(n+\frac{1}{2})} - \underline{\alpha}') + l(\beta_{(n+\frac{1}{2})} - \underline{\alpha}).$$

Adding and subtracting $f(\xi, \beta_{(n+\frac{1}{2})}, \underline{\alpha}')$ on right hand side we get

$$-(\beta''_{(n+\frac{3}{2})} - \underline{\alpha}'') + \sqrt[3]{l}K(\xi)(\beta'_{(n+\frac{3}{2})} - \underline{\alpha}') + l(\beta_{(n+\frac{3}{2})} - \underline{\alpha}) \geq -f(\xi, \beta_{(n+\frac{1}{2})}, \beta'_{(n+\frac{1}{2})}) + f(\xi, \beta_{(n+\frac{1}{2})}, \underline{\alpha}') - f(\xi, \beta_{(n+\frac{1}{2})}, \underline{\alpha}') + f(\xi, \underline{\alpha}, \underline{\alpha}') + \sqrt[3]{l}K(\xi)(\beta'_{(n+\frac{1}{2})} - \underline{\alpha}') + l(\beta_{(n+\frac{1}{2})} - \underline{\alpha}).$$

Since $\underline{\alpha} \leq \beta_{(n+\frac{1}{2})}$ then by **[C2]** and **[C3]** we can write

$$-w_1'' + \sqrt[3]{l}K(\xi)w_1' + lw_1 \geq -N \mid \beta'_{(n+\frac{1}{2})} - \underline{\alpha}' \mid - M(\beta_{(n+\frac{1}{2})} - \underline{\alpha}) + \sqrt[3]{l}K(\xi)w_2' + lw_2,$$

or

$$-w_1'' + \sqrt[3]{l}K(\xi)w_1' + lw_1 \geq -N \mid w_2' \mid - Mw_2 + \sqrt[3]{l}K(\xi)w_2' + lw_2,$$

where $w_1 = \beta_{(n+\frac{3}{2})} - \underline{\alpha}$ and $w_2 = \beta_{(n+\frac{1}{2})} - \underline{\alpha}$. Multiplying both sides by -1 and using the fact that $\mid w_2' \mid = \text{sign}(w_2')w_2'$ we obtain

$$w_1'' - \sqrt[3]{l}K(\xi)w_1' - lw_1 \leq (M-l)w_2 + (N\text{sign}(w_2') - \sqrt[3]{l}K(\xi))w_2'. \tag{4.50}$$

By Lemma3.2.13we conclude from (4.50) that

$$w_1'' - \sqrt[3]{l}K(\xi)w_1' - lw_1 \leq 0.$$

As $\underline{\alpha} \leq \beta_{(n+\frac{3}{2})}$ on $\partial\Omega$, by Lemma3.2.4we conclude that $\underline{\alpha} \leq \beta_{(n+\frac{3}{2})}$ on Ω. Hence $\underline{\alpha} \leq \beta_{(n+\frac{1}{2})}$ for all $n \in \mathcal{N}$. This completes the proof of (4.43). ■

Theorem 4.2.5 *Let $\mathcal{X} = \mathcal{C}^1([a,b])$, $\mathcal{Z} = \mathcal{C}([a,b])$ also let α and $\beta \in \mathcal{X}$ with $\alpha \leq \beta$. Let*

$$\epsilon = \{u \in \mathcal{X} | \alpha \leq u \leq \beta\}$$

and $T : \epsilon \to \mathcal{X}$ be the continuous operator defined in (3.56). The operator T satisfies $\alpha \leq T(\alpha)$ and $\beta \geq T(\beta)$. Then the sequences $\{\alpha_{(n+\frac{1}{2})}\}$ defined by $\alpha_{(-\frac{1}{2})} = \alpha_{(0)} = \alpha$, $\alpha_{(n+\frac{1}{2})} = T(\alpha_{(n-\frac{1}{2})})$ on Ω_1, converge monotonically in $\mathcal{X}$ to a fixed point $u^1_{\min}$ of T on Ω_1 and the sequences $\{\alpha_{(n+1)}\}$ defined by $\alpha_{(0)} = \alpha$, $\alpha_{(n+1)} = T(\alpha_{(n)})$ on Ω_2 also converge monotonically in $\mathcal{X}$ to a fixed point $u^2_{\min}$ of T on Ω_2.

Proof: On Ω_1 we claim that the sequence $\{\alpha_{(n+\frac{1}{2})}\}$ converges in $\mathcal{X}$. We deduce from the monotonicity of T that the sequence $\{\alpha_{(n+\frac{1}{2})}\}$ is increasing and bounded;

$$\alpha_{(0)} = \alpha_{(-\frac{1}{2})} \leq T(\alpha_{(-\frac{1}{2})}) = \alpha_{(\frac{1}{2})} \leq T(\alpha_{(\frac{1}{2})}) = \alpha_{(\frac{3}{2})} \leq ... \quad ... \leq T(\beta) \leq \beta.$$

Hence $\{\alpha_{(n+\frac{1}{2})}\}$ is included in ϵ. As the sequence is monotone and bounded, the point wise limit exists, that is

$$\lim_{n\to\infty} \alpha_{(n+\frac{1}{2})}(\xi) = u^1_{\min}(\xi).$$

By Lemma3.2.19we know that there exists a positive constant μ_1 such that

$$| \alpha'_{(n+\frac{1}{2})} | \leq \mu_1 \text{ on } \partial\Omega_1.$$

Hence by Lemma3.2.14we conclude that first derivative of the sequence is bounded. That is there exists $R \geq 0$ such that $| \alpha'_{(n+\frac{1}{2})} |_\infty < R$. This implies for all n

$$\left| \frac{\alpha_{(n+\frac{1}{2})}(\xi_1) - \alpha_{(n+\frac{1}{2})}(\xi_2)}{\xi_1 - \xi_2} \right| < R,$$

as $\xi_1 \to \xi_2$, where $\xi_1, \xi_2 \in \Omega_1$. This implies the sequence of functions $\alpha_{(n+\frac{1}{2})}$ has a Lipschitz constant which implies that the sequence is equicontinuous. Hence by

Theorem3.1.9there exist a subsequence $\alpha_{(n+\frac{1}{2})_k}$ of $\alpha_{(n+\frac{1}{2})}$ which converges in $\mathcal{X}$ and therefore in $\mathcal{Z}$. By uniqueness of the limit and the monotonicity of the sequence $\alpha_{(n+\frac{1}{2})}$ we conclude that $\alpha_{(n+\frac{1}{2})} \to u^1_{\min}$ in $\mathcal{X}$ on Ω_1. We also claim that $u^1_{\min}$ is a fixed point of T on Ω_1. We notice that

$$\lim_{n\to\infty} T(\alpha_{(n+\frac{1}{2})}) = \lim_{n\to\infty} \alpha_{(n+\frac{3}{2})} = u^1_{\min}.$$

As T is continuous we have

$$T(u^1_{\min}) = u^1_{\min}.$$

On Ω_2 we claim that the sequence $\{\alpha_{(n+1)}\}$ converges in $\mathcal{X}$. From the monotonicity of T we know that the sequence $\{\alpha_{(n+1)}\}$ is increasing and bounded;

$$T(\alpha_{(0)}) = \alpha_{(1)} \leq T(\alpha_{(1)}) = \alpha_{(2)} \leq ... \quad ... \leq T(\beta) \leq \beta.$$

Hence $\{\alpha_{(n+1)}\}$ is included in ϵ. As the sequence is monotone and bounded, the point wise limit exists, that is

$$\lim_{n\to\infty} \alpha_{(n+1)}(\xi) = u^2_{\min}(\xi).$$

By Lemma3.2.19we know that there exists a positive constant μ_2 such that

$$\mid \alpha^{'}_{(n+1)} \mid \leq \mu_2 \text{ on } \partial\Omega_2.$$

Hence by Lemma3.2.14we conclude that first derivative of the sequence is bounded. That is there exists $R \geq 0$ such that $\mid \alpha^{'}_{(n+1)} \mid_\infty < R$ This implies for all n,

$$\left| \frac{\alpha_{(n+1)}(\xi_1) - \alpha_{(n+1)}(\xi_2)}{\xi_1 - \xi_2} \right| < R,$$

as $\xi_1 \to \xi_2$, where $\xi_1, \xi_2 \in \Omega_2$. This implies the sequence of functions $\alpha_{(n+1)}$ has a Lipschitz constant which implies that the sequence is equicontinuous. Hence by

Theorem3.1.9there exist a subsequence $\alpha_{(n+1)_k}$ of $\alpha_{(n+1)}$ which converges in $\mathcal{X}$ and therefore in $\mathcal{Z}$. By uniqueness of the limit and the monotonicity of the sequence $\alpha_{(n+1)}$ we conclude that $\alpha_{(n+1)} \to u^2_{\min}$ in $\mathcal{X}$ on Ω_2. We also claim that $u^2_{\min}$ is a fixed point of T on Ω_2. We notice that

$$\lim_{n\to\infty} T(\alpha_{(n+1)}) = \lim_{n\to\infty} \alpha_{(n+2)} = u^2_{\min}.$$

As T is continuous we have

$$T(u^2_{\min}) = u^2_{\min}.$$

From Lemma4.2.3we conclude that as $n \to \infty$, $u^1_{\min} = u^2_{\min} = u^*$ on Ω. Hence the theorem is proved. ■

Theorem 4.2.6 *Let $\mathcal{X} = \mathcal{C}^1([a,b])$, $\mathcal{Z} = \mathcal{C}([a,b])$ also let α and $\beta \in \mathcal{X}$, $\alpha \leq \beta$. Let*

$$\epsilon = \{u \in \mathcal{X} | \alpha \leq u \leq \beta\}$$

and $T : \epsilon \to \mathcal{X}$ be the continuous operator defined in (3.56). The operator T satisfies $\alpha \leq T(\alpha)$ and $\beta \geq T(\beta)$. Then the sequences $\{\beta_{(n+\frac{1}{2})}\}$ defined by $\beta_{(-\frac{1}{2})} = \beta_{(0)} = \beta$, $\beta_{(n+\frac{1}{2})} = T(\beta_{(n-\frac{1}{2})})$ on Ω_1 converges monotonically in $\mathcal{X}$ to a fixed point $u^1_{\max}$ of T on Ω_1 and the sequences $\{\beta_{(n+1)}\}$ defined by $\beta_{(0)} = \beta$, $\beta_{(n+1)} = T(\beta_{(n)})$ on Ω_2 also converges monotonically in $\mathcal{X}$ to a fixed point $u^2_{\max}$ of T on Ω_2.

Proof: On Ω_1 we claim that the sequence $\{\beta_{(n+\frac{1}{2})}\}$ converges in $\mathcal{X}$. We deduce from the monotonicity of T that the sequence $\{\beta_{(n+\frac{1}{2})}\}$ is decreasing and bounded;

$$\beta_{(0)} = \beta_{(-\frac{1}{2})} \geq T(\beta_{(-\frac{1}{2})}) = \beta_{(\frac{1}{2})} \geq ... \quad ... \geq T(\alpha) \geq \alpha.$$

Hence $\{\beta_{(n+\frac{1}{2})}\}$ is included in ϵ. As the sequence is monotone and bounded, the point wise limit exists, that is

$$\lim_{n\to\infty} \beta_{(n+\frac{1}{2})}(\xi) = u^1_{\max}(\xi).$$

By Lemma3.2.19we can say that there exists a positive constant μ_3 such that

$$| \beta'_{(n+\frac{1}{2})} |\leq \mu_3 \text{ on } \Omega_1.$$

Hence by Lemma3.2.15we conclude that first derivative of the sequence is bounded. That is there exists $R \geq 0$ such that $| \beta'_{(n+\frac{1}{2})} |_\infty < R$. This implies for all n

$$\left| \frac{\beta_{(n+\frac{1}{2})}(\xi_1) - \beta_{(n+\frac{1}{2})}(\xi_2)}{\xi_1 - \xi_2} \right| < R,$$

as $\xi_1 \to \xi_2$, where $\xi_1, \xi_2 \in \Omega_1$. This implies the sequence of functions $\beta_{(n+\frac{1}{2})}$ has a Lipschitz constant which implies that the sequence is equicontinuous. Hence by Theorem3.1.9there exist a subsequence $\beta_{(n+\frac{1}{2})_k}$ of $\beta_{(n+\frac{1}{2})}$ which converges in $\mathcal{X}$ and therefore in $\mathcal{Z}$. By uniqueness of the limit and the monotonicity of the sequence $\beta_{(n+\frac{1}{2})}$ we conclude that $\beta_{(n+\frac{1}{2})} \to u^1_{\max}$ in $\mathcal{X}$ on Ω_1. We also claim that $u^1_{\max}$ is a fixed point of T on Ω_1. We notice that

$$\lim_{n\to\infty} T(\beta_{(n+\frac{1}{2})}) = \lim_{n\to\infty} \beta_{(n+\frac{3}{2})} = u^1_{\max}.$$

As T is continuous we have

$$T(u^1_{\max}) = u^1_{\max}.$$

On Ω_2 we also claim that the sequence $\{\beta_{(n+1)}\}$ converges in $\mathcal{X}$. We deduce from the monotonicity of T that the sequence $\{\beta_{(n+1)}\}$ is decreasing and bounded;

$$T(\beta_{(0)}) = \beta_{(1)} \geq T(\beta_{(1)}) = \beta_{(2)} \geq ... \quad ... \geq T(\alpha) \geq \alpha,$$

Hence $\{\beta_{(n+1)}\}$ is included in ϵ. As the sequence is monotone and bounded, the point wise limit exists, that is

$$\lim_{n\to\infty} \beta_{(n+1)}(\xi) = u^2_{\max}(\xi).$$

By Lemma3.2.19we can say that there exists a positive constant μ_4 such that

$$| \beta'_{(n+1)} | \leq \mu_4 \text{ on } \Omega_2.$$

Hence by Lemma3.2.15we conclude that first derivative of the sequence is bounded. That is there exists $R \geq 0$ such that $| \beta'_{(n+1)} |_\infty < R$. This implies for all n

$$\left| \frac{\beta_{(n+1)}(\xi_1) - \beta_{(n+1)}(\xi_2)}{\xi_1 - \xi_2} \right| < R,$$

as $\xi_1 \to \xi_2$, where $\xi_1, \xi_2 \in \Omega_2$. This implies the sequence of functions $\beta_{(n+1)}$ has a Lipschitz constant which implies that the sequence is equicontinuous. Hence by Theorem3.1.9there exist a subsequence $\beta_{(n+1)_k}$ of $\beta_{(n+1)}$ which converges in $\mathcal{X}$ and therefore in $\mathcal{Z}$. By uniqueness of limit and monotonicity of the sequence $\beta_{(n+1)}$ we conclude that $\beta_{(n+1)} \to u^2_{\max}$ in $\mathcal{X}$ on Ω_2. We also claim that $u^2_{\max}$ is a fixed point of T on Ω_2. We notice that

$$\lim_{n\to\infty} T(\beta_{(n+1)}) = \lim_{n\to\infty} \beta_{(n+2)} = u^2_{\max}.$$

As T is continuous we have

$$T(u^2_{\max}) = u^2_{\max}.$$

From Lemma4.2.4we conclude that as $n \to \infty$, $u^1_{\max} = u^2_{\max} = u^*$ on Ω. Hence the theorem is proved. ■

Remark 2 *If the BVP has unique solution then sequence of subsolutions and supersolutions converge to the true solution.*

In the next chapter we will provide the numerical results of all the theory we have discussed.

Chapter 5

Numerical Results

This chapter illustrates numerical results of the theory we have discussed in this thesis. Section5.1gives numerical results for nonlinear domain decomposition. In Section5.2we provide numerical examples for linearized single domain approaches for BVPs. Finally in Section5.3numerical examples for linearized domain decomposition approaches are demonstrated.

5.1 Nonlinear Domain Decomposition Method

Now we will solve the mesh BVP

$$\frac{d}{d\xi}\left\{M(x(\xi),u)\frac{d}{d\xi}x(\xi)\right\} = 0 \tag{5.1}$$

$$x(0) = 0, \quad x(1) = 1.$$

for two subdomains and several subdomains. Here we consider the monitor function $M(x(\xi)) = 1 + x(\xi)^2$.

5.1.1 Nonlinear DD for two subdomains

Suppose the domain is decomposed into two overlapping subdomains. In order to see the effect of the overlap on the convergence we plot the DD error against the number of iterations. The DD error is defined as the infinity norm of the error between the global numerical solution and the DD solution in the subdomain.

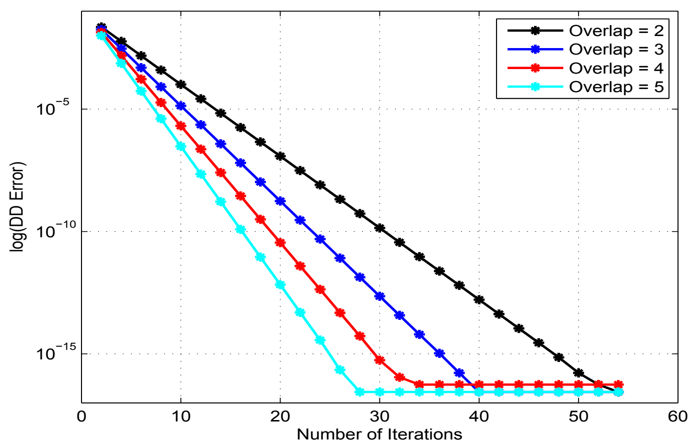


Figure 5.1: DD error vs number of iterations for different overlap for the 1st subdomain for BVP (5.1).

Figure5.1shows the DD error for different overlaps on 1st subdomain and Figure5.2 shows the result on the 2nd subdomain. We notice that the larger the overlap, the faster the DD error decreases.

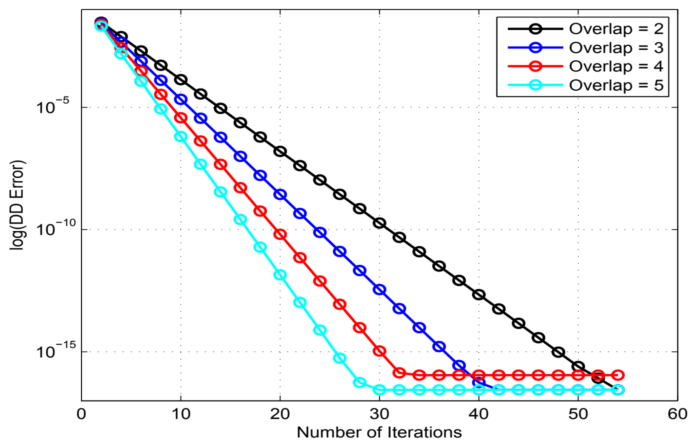


Figure 5.2: DD error vs number of iterations for different overlap for the 2nd subdomain for BVP (5.1).

5.1.2 Nonlinear DD for several subdomains

Now we decompose the domain into several subdomains. To see the effect of the number of subdomains on the convergence we plot the DD error against the number of iterations for different numbers of subdomain in Figure5.3.

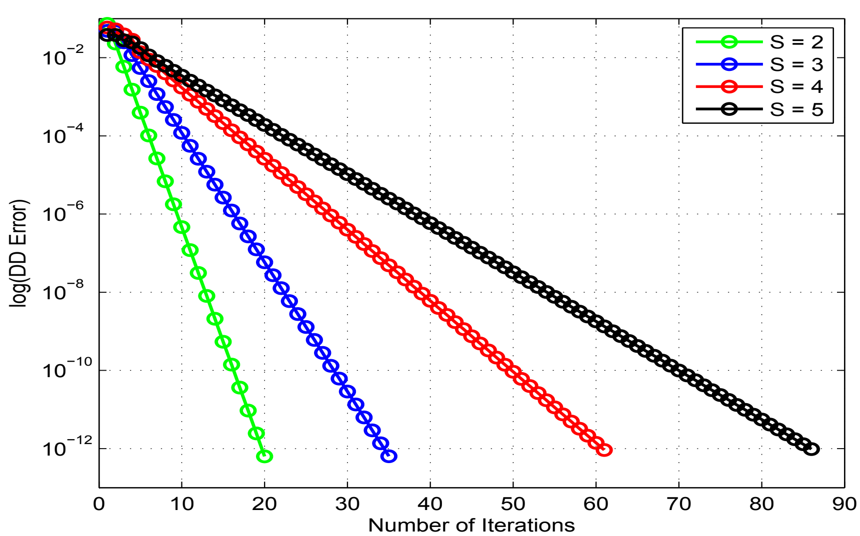


Figure 5.3: DD error vs iteration for different numbers of subdomains for BVP (5.1).

We certainly observe from Figure5.3that for fewer numbers of subdomains the DD solution converges more quickly.

5.2 Numerical Results for the linearized Single Domain methods

5.2.1 Numerical results for a linearized single domain method to solve $u^{''} = f(\xi, u)$

To illustrate Theorem3.1.11here we will provide some numerical experiments.

In all the numerical examples the number of grid points we have considered is 321 and the iterations will stop when the difference in the infinity norm between two successive iterates is below the predefined tolerance of 10^{-6}. In the iterations for problems of the form $u^{''} = f(\xi, u, u^{'})$ we have used the function $K(\xi) = \frac{1}{2} - \xi$ which is anti-symmetric and satisfies $K(0) > 0$ as required in the theory. To obtain a subsolution for a given problem, we construct a polynomial and then choose the coefficients of the polynomial satisfying the properties of a subsolution. Like wise we construct the supersolution of the given problem.

Example 5.2.1.1 Consider the BVP on $\Omega = [0, 1]$ from [13]

$$u^{''}(\xi) = -20 + 60\pi\xi\cos(20\pi\xi^3) - \frac{(60\pi\xi^2)^2}{2}\sin(20\pi\xi^3), \qquad u = 0 \text{ on } \partial\Omega. \tag{5.2}$$

The analytic solution of the BVP (5.2) is given by,

$$u(\xi) = 10\xi - 10\xi^2 + \frac{1}{2}\sin(20\pi\xi^3),$$

which is plotted in Figure5.4below.

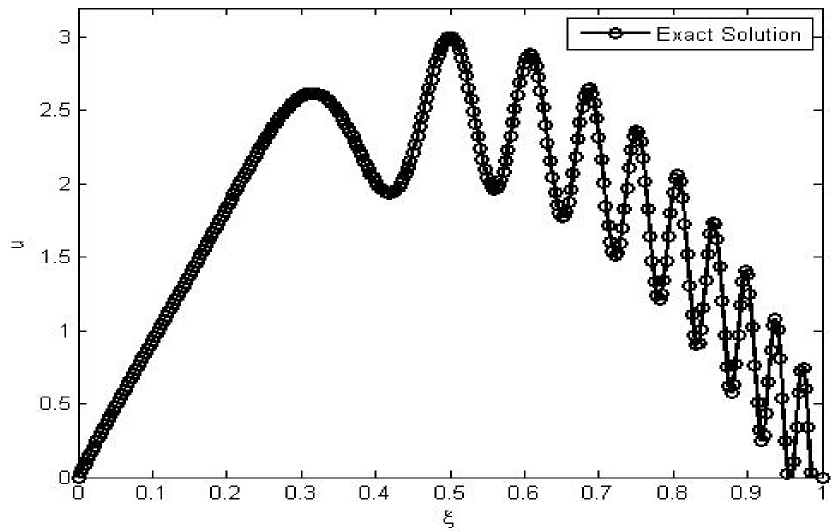


Figure 5.4: Plot of the exact solution of BVP (5.2).

A subsolution of BVP (5.2) is

$$\underline{u}(\xi) = \sum_{i=1}^{5} (-1)^i a_i \xi^{5-(i-1)}, \tag{5.3}$$

and a supersolution is

$$\bar{u}(\xi) = \sum_{i=1}^{5} (-1)^{i+1} a_i \xi^{5-(i-1)}, \tag{5.4}$$

where the constant coefficients are $a_1 = 15.8730$, $a_2 = 117.40603$, $a_3 = 173.1746$, $a_4 = 131.9683$ and $a_5 = 60.3810$. The numerical solution of BVP (5.2) using the linearized iterations stated in Theorem3.1.11starting from subsolution and supersolution are presented in Figure5.5and5.6respectively.

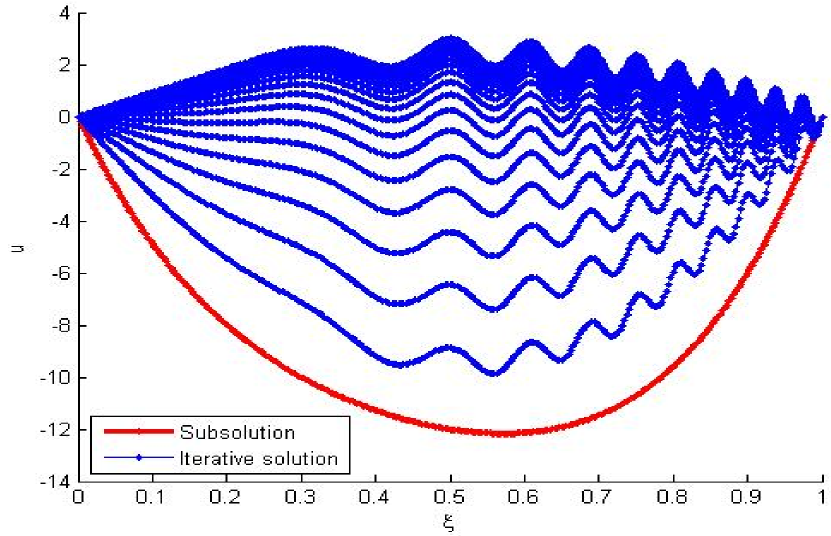


Figure 5.5: Linearized iterations starting from the subsolution for BVP (5.2).

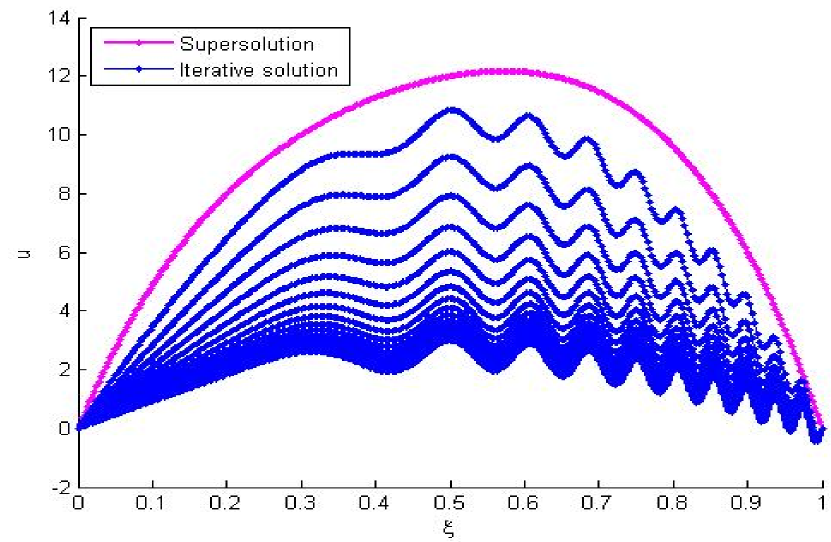
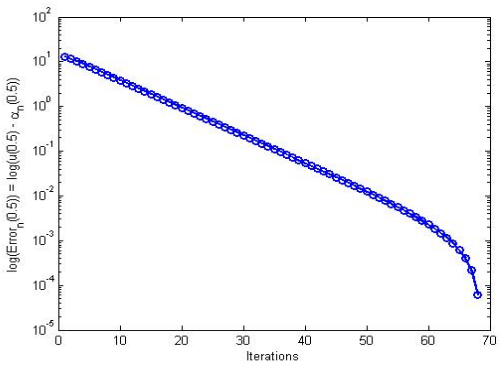
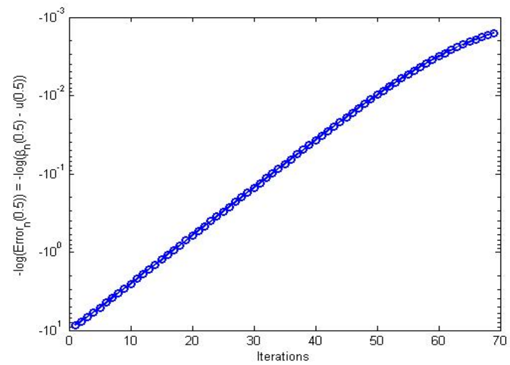


Figure 5.6: Linearized iterations starting from the supersolution for BVP (5.2).

Figure5.7aillustrates that iterations starting from subsolution are increasing monotonically, that is error tends to zero from a positive value. Likewise Figure5.7b shows that iterations starting from supersolution are decreasing monotonically that is error tends to zero from a negative value, when the error is calculated at a single point. Both of these iterations are converging to the true solution.



(a) Iterations starting from subsolution.



(b) Iterations starting from supersolution.

Figure 5.7: Monotonicity of the iterates for BVP (5.2).

Example 5.2.1.2 Consider the BVP,

$$-u^{''} = u - 3\xi - 5\tan(\xi), \qquad u(0) = 0 \qquad u(1) = 1. \tag{5.5}$$

The analytic solution of the BVP (5.5) is given by,

$$u(\xi) = \frac{\sin(\xi)\{5\ln\left(\frac{1+\sin(1)}{\cos(1)}\right)\cos(1) - 2\}}{\sin(1)} + 3\xi - 5\ln\left\{\frac{1+\sin(\xi)}{\cos(\xi)}\right\}\cos(\xi).$$

Figure5.8shows the analytic solution of equation (5.5).

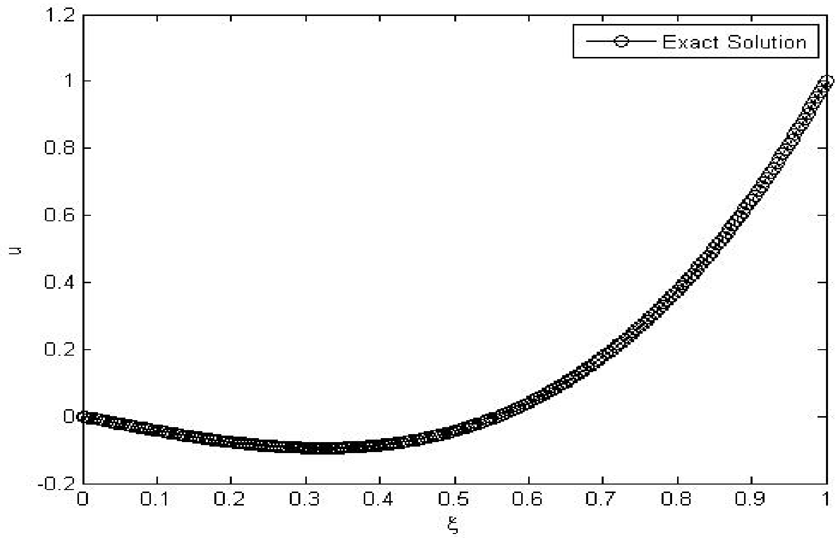


Figure 5.8: Plot of the analytic solution of BVP (5.5).

A subsolution for BVP (5.5) is

$$\underline{u}(\xi) = 10\xi^2 - 9\xi, \tag{5.6}$$

and a supersolution for BVP (5.5) is

$$\bar{u}(\xi) = -2\xi^2 + 3\xi. \tag{5.7}$$

Figures5.9and5.10show the numerical solution of BVP (5.5) using iterations stated in Theorem3.1.11starting from the subsolution and suppersolution respectively.

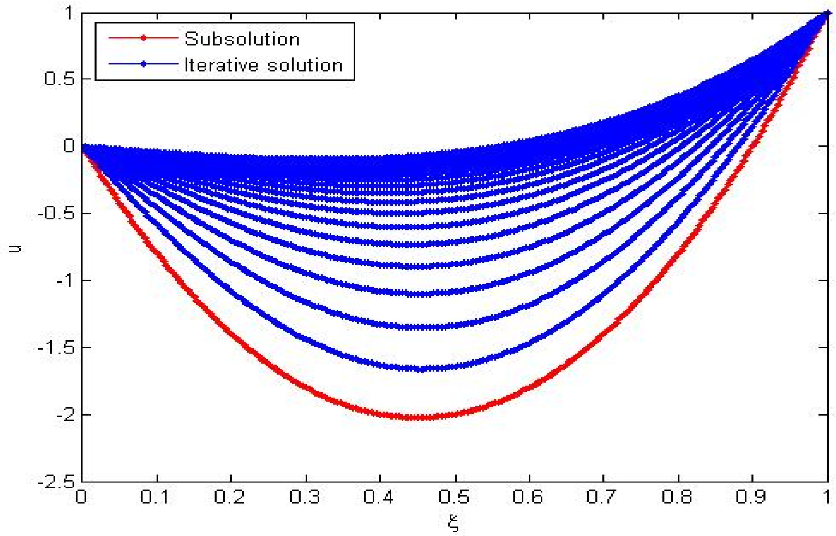


Figure 5.9: Linearized Iterations starting from the subsolution for BVP (5.5).

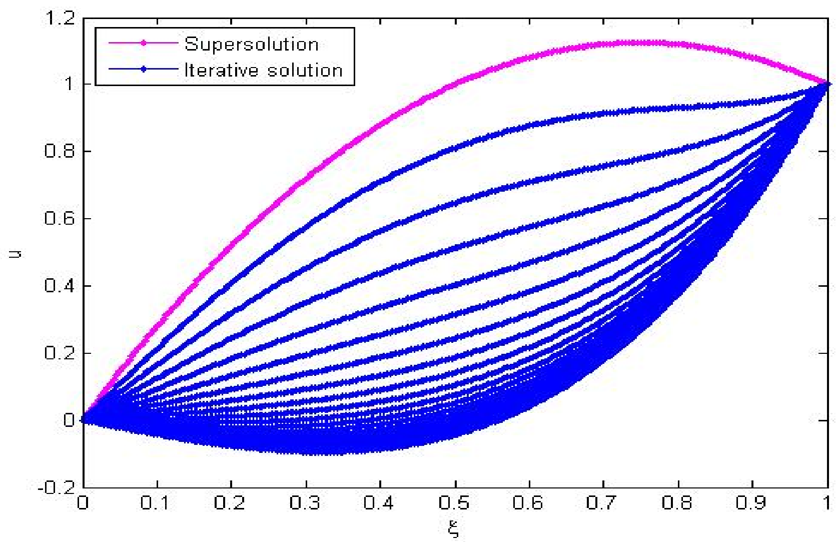
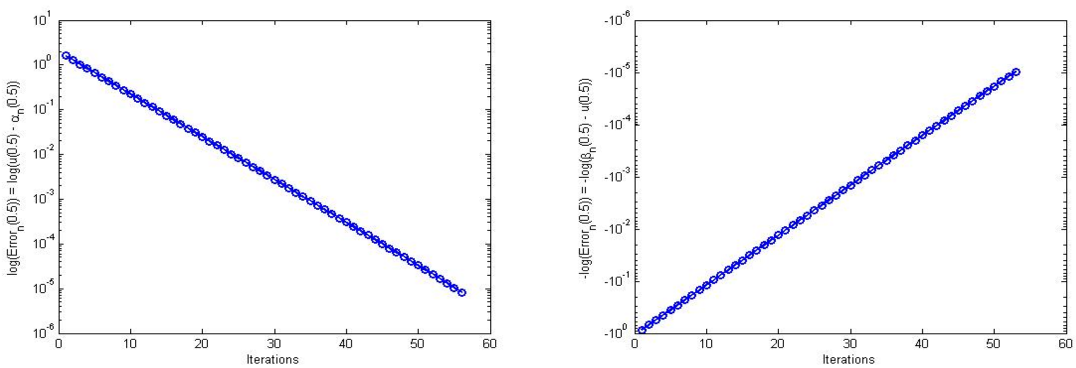


Figure 5.10: Linearized iterations starting from the supersolution for BVP (5.5).

Figure5.11aindicates that iterations starting from subsolution increases monotonically, that is error tends to zero from a positive value. Similarly Figure5.11b shows that iterations starting from supersolution decreases monotonically, that is error tends to zero from a negative value, where error is calculated at a single point. Both of these iterations are converging to the global numerical solution.



(a) Iterates starting from subsolution. (b) Iterates starting from supsolution.

Figure 5.11: Monotonicity of iterations for the solution of BVP (5.5).

Example 5.2.1.3 Consider the nonlinear BVP,

$$u^{''} = e^u, \qquad u(0) = 0 \qquad u(1) = 1. \tag{5.8}$$

The analytic solution of the BVP (5.8) is given by,

$$u(\xi) = -\log(2) + 2\log[a\sec(\frac{a}{2}(\xi - \frac{1}{2}))],$$

where $a = 1.3306557$. Figure5.12shows the analytic solution of equation (5.8).

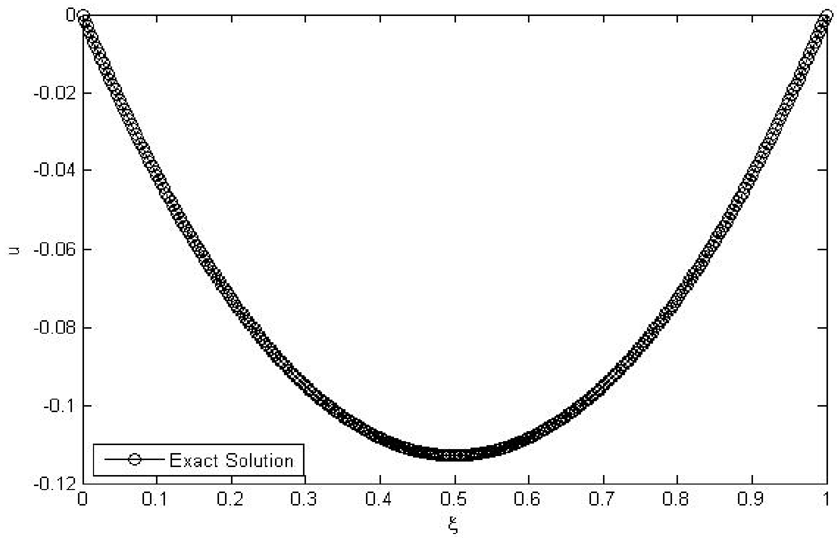


Figure 5.12: Plot of the analytic solution of BVP (5.5).

A subsolution for BVP (5.8) is

$$\underline{u}(\xi) = \xi(\xi - 1), \tag{5.9}$$

and a supersolution for BVP (5.8) is

$$\bar{u}(\xi) = -\xi(\xi - 1). \tag{5.10}$$

Figures5.13and5.14show the numerical solution of BVP (5.8) using iterations stated in Theorem3.1.11starting from the subsolution and suppersolution respectively.

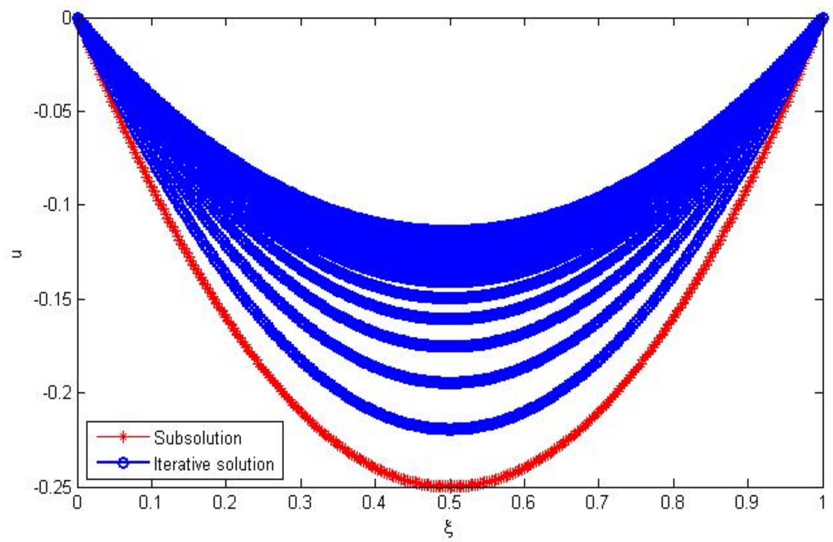


Figure 5.13: Linearized Iterations starting from the subsolution for BVP (5.8).

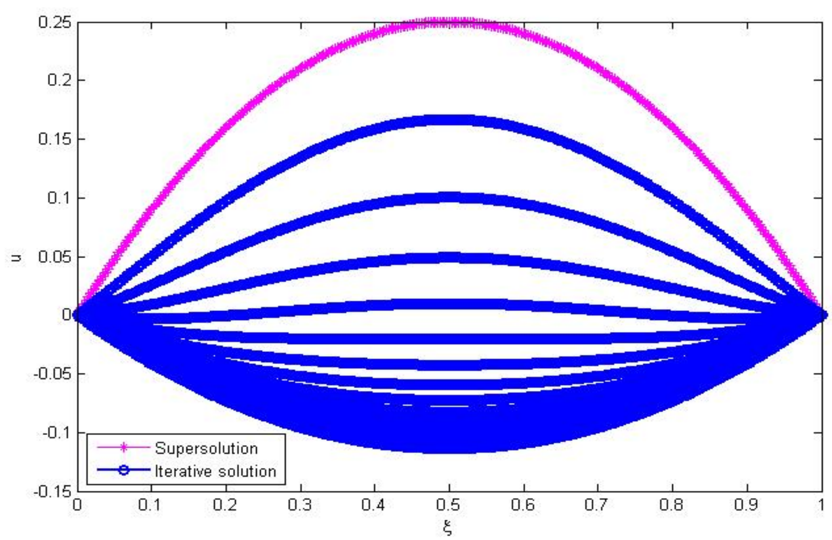
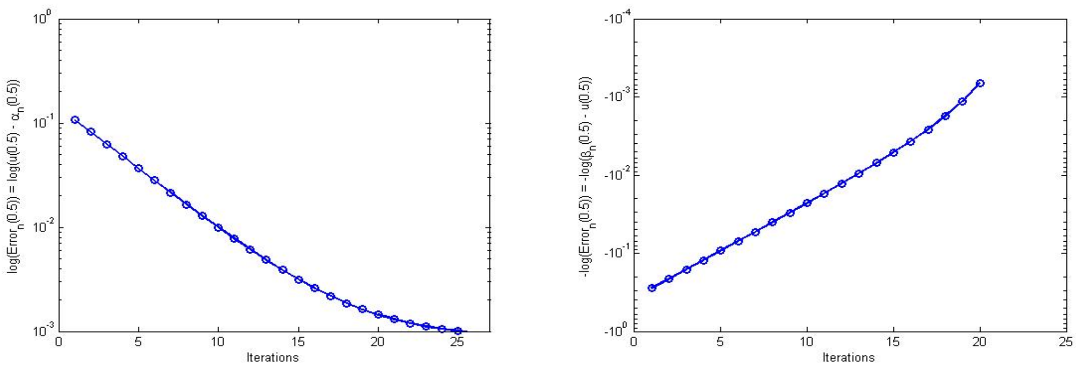


Figure 5.14: Linearized iterations starting from the supersolution for BVP (5.8).

Figure5.15aindicates that iterations starting from subsolution increases monotonically, that is error tends to zero from a positive value. Similarly Figure5.15b shows that iterations starting from supersolution decreases monotonically, that is error tends to zero from a negative value, where error is calculated at a single point. Both of these iterations are converging to the global numerical solution.



(a) Iterates starting from subsolution. (b) Iterates starting from supsolution.

Figure 5.15: Monotonicity of iterations for the solution of BVP (5.8).

5.2.2 Numerical results for the Linearized single domain method to solve $u^{''} = f(\xi, u, u^{'})$

To illustrate Theorem3.2.22and3.2.23we will provide some numerical experiments.

Example 5.2.2.1 We first consider a BVP on $\Omega = [0, 1]$ for which f has no $u^{'}$ dependence.

$$u^{''}(\xi) = -20 + 60\pi\xi\cos(20\pi\xi^3) - \frac{(60\pi\xi^2)^2}{2}\sin(20\pi\xi^3), \qquad u = 0 \text{ on } \partial\Omega. \quad (5.11)$$

The exact solution of the BVP (5.11) is given by

$$u(\xi) = 10\xi - 10\xi^2 + \frac{1}{2}\sin(20\pi\xi^3),$$

which is plotted in Figure5.4. The numerical solution of BVP (5.11) obtained from the iterations (3.19) starting from the subsolution (5.3) is presented in Figure5.16.

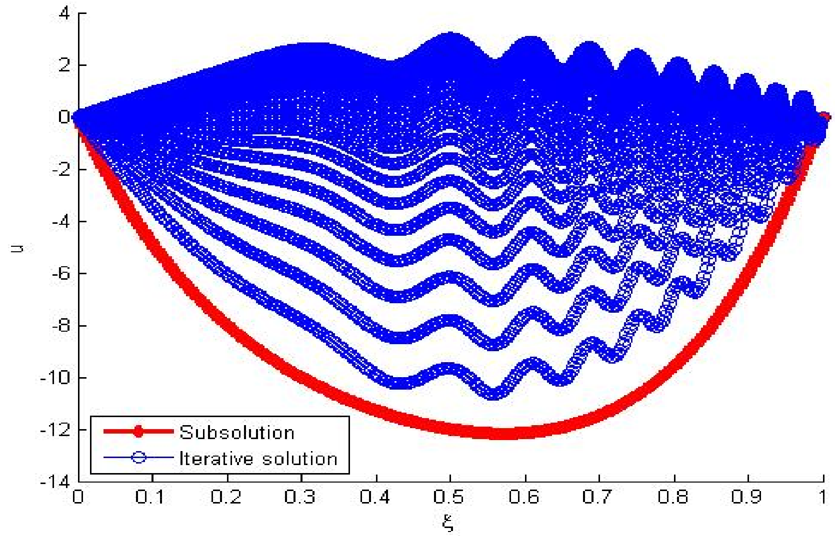


Figure 5.16: Linearized iterations starting from the subsolution for BVP (5.11).

Likewise the numerical solution of BVP (5.11) obtained from the iterations (3.20) starting from the supersolution (5.4) is shown in Figure5.17.

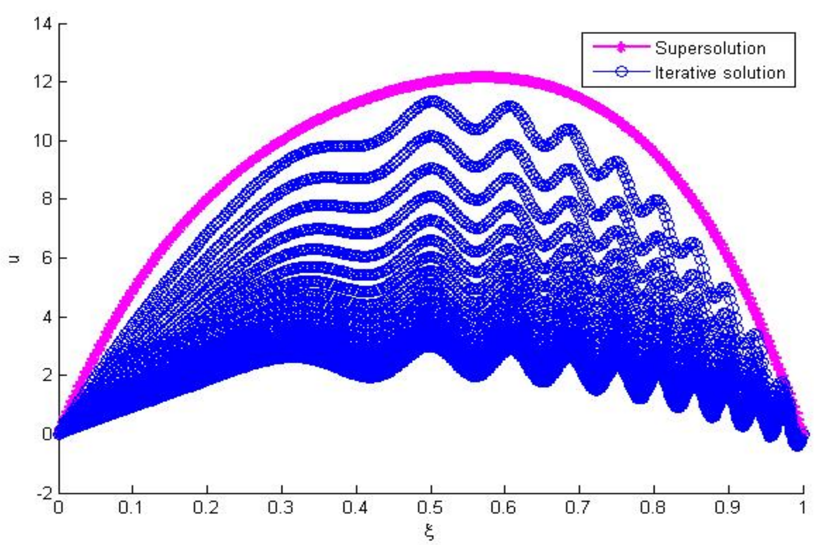
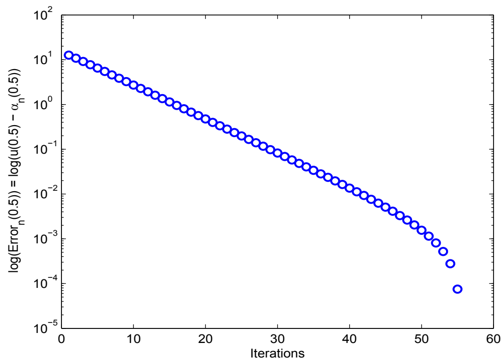
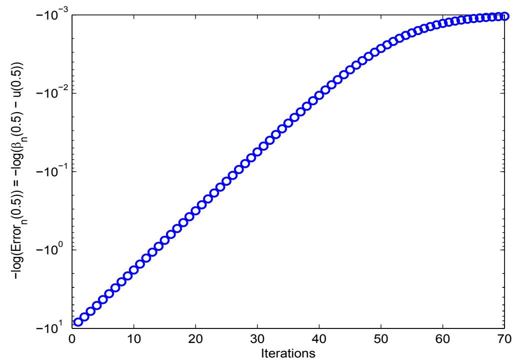


Figure 5.17: Linearized iterations starting from the supersolution for BVP (5.11).

We observe from Figure5.18aand5.18bthat the iterations starting from the subsolution increases monotonically and the iteration starting from the supersolution decreases monotonically to the true solution.



(a) Solution increasing monotonically.



(b) Solution decreasing monotonically.

Figure 5.18: Monotonicity of iterations for BVP (5.11).

Example 5.2.2.2 Consider another BVP

$$-u^{''} = u^{'} - 3x, \qquad u(0) = 0, \quad u(1) = 0. \tag{5.12}$$

whose exact solution is given by

$$u(\xi) = \frac{3}{2}x^2 + \frac{3e^{-x}}{2(e^{-1}-1)} - 3x - \frac{3}{2(e^{-1}-1)}.$$

Figure5.19shows the exact solution of BVP (5.12).

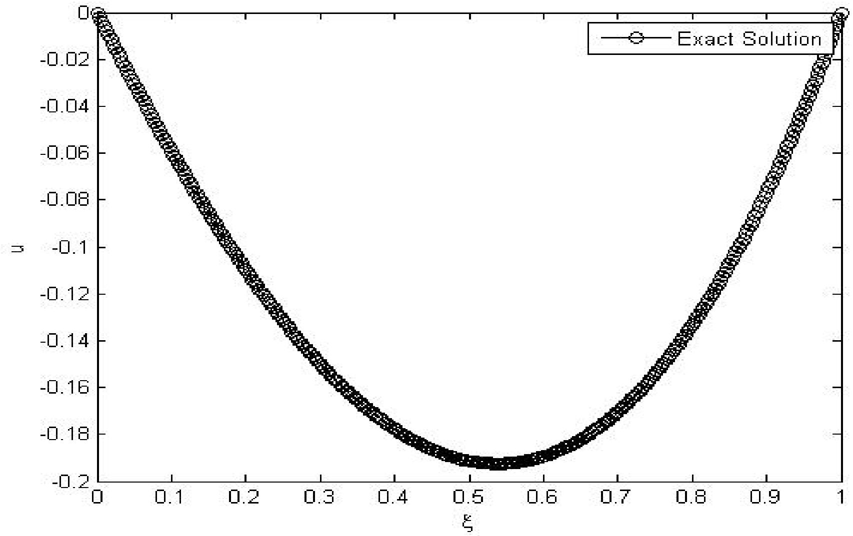


Figure 5.19: Plot of the analytic solution of BVP (5.12).

A subsolution for BVP (5.12) is

$$\underline{u}(\xi) = \xi(\xi - 1), \tag{5.13}$$

and a supersolution for BVP (5.12) is

$$\bar{u}(\xi) = \frac{1}{2}\xi(\xi - 1). \tag{5.14}$$

Figure5.20and5.21shows the numerical solution of BVP (5.12) using Cherpion's single domain iterations starting from the subsolution and supersolution respectively.

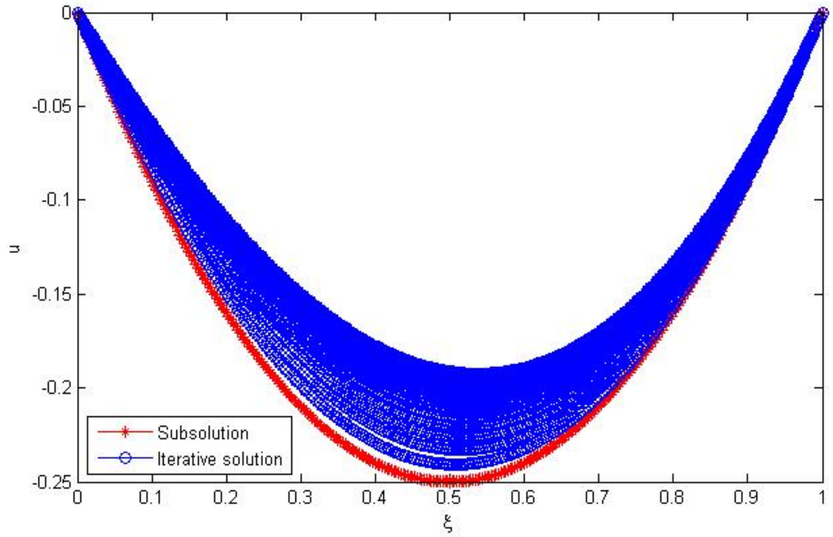


Figure 5.20: Linearized iterations starting from the subsolution for BVP (5.12).

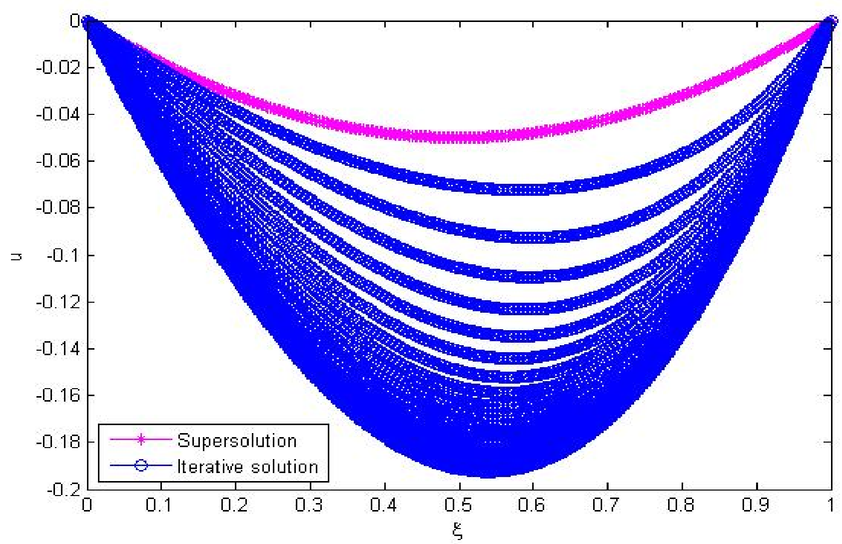
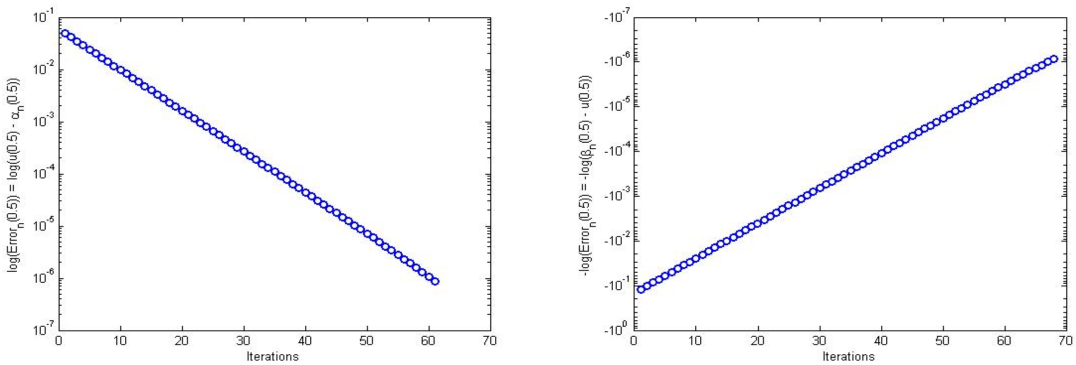


Figure 5.21: Linearized iterations starting from the supersolution for BVP (5.12).

We observe from Figure5.22aand5.22bthat the solution starting from the subsolution increases monotonically and the solution starting from the supersolution decreases monotonically to the true solution.



(a) Monotonicity of solution starting from the subsolution.

(b) Monotonicity of solution starting from the supsolution.

Figure 5.22: Monotonicity of iterations for BVP (5.12).

Example 5.2.2.3 Consider the nonlinear BVP

$$u^{''} = \sin(u^{'}) + 2 - \sin(2\xi - 1), \qquad u(0) = 0, \quad u(1) = 0. \tag{5.15}$$

Whose exact solution is given by

$$u(\xi) = \xi^2 - \xi.$$

Figure5.23shows the exact solution of equation (5.15).

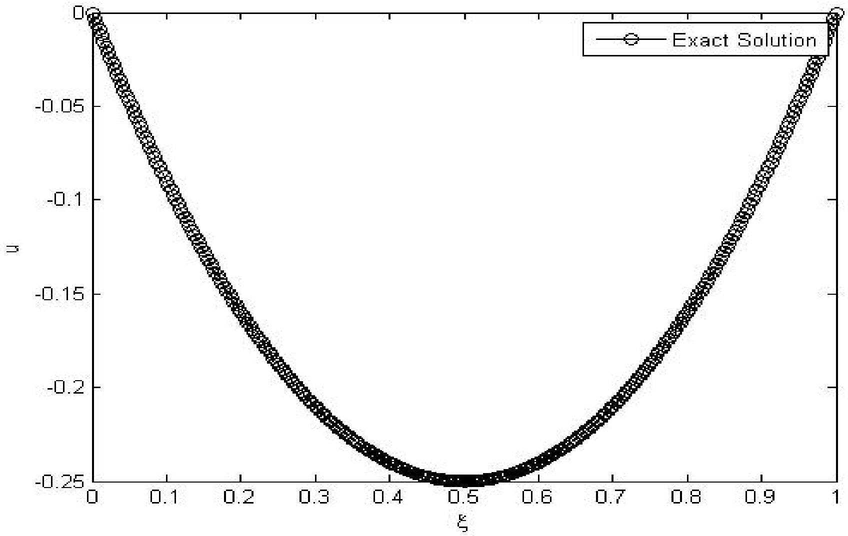


Figure 5.23: Plot of the analytic solution for BVP (5.15).

BVP (5.15) satisfies **[C2]** and **[C3]**, the assumptions of the Theorem3.2.22and 3.2.23. A subsolution for BVP (5.15) is

$$\underline{u}(\xi) = \frac{5}{2}\xi(\xi - 1), \tag{5.16}$$

and a supersolution is

$$\bar{u}(\xi) = -\frac{5}{2}\xi(\xi - 1). \tag{5.17}$$

Figure5.24and5.25shows the numerical solution of BVP (5.15) using Cherpion's single domain iterations starting from the subsolution and supersolution respectively.

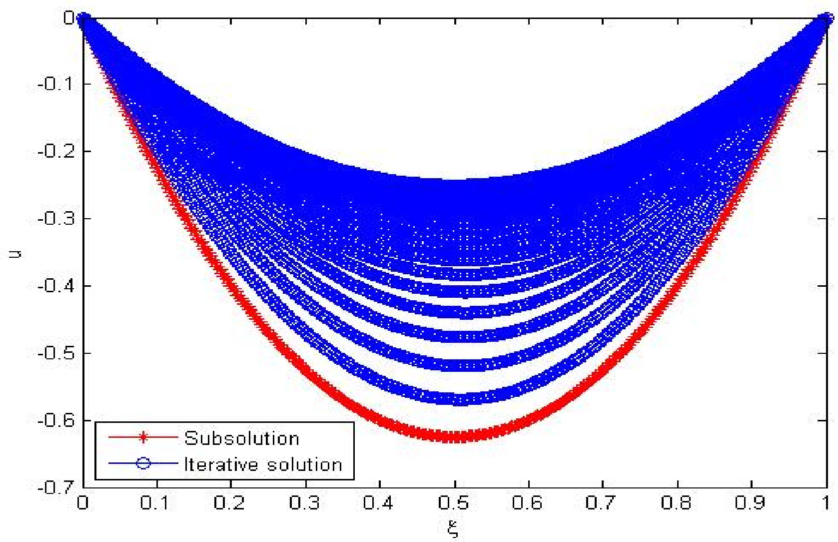


Figure 5.24: Linearized iterations starting from the subsolution for BVP (5.15).

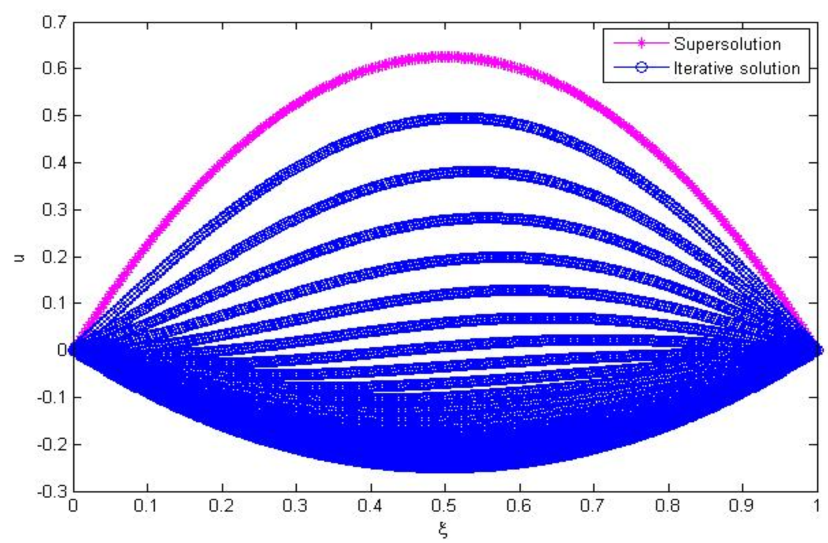
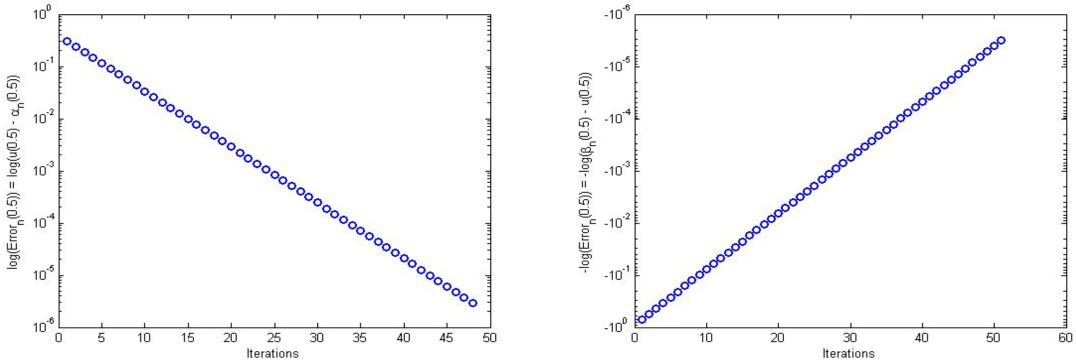


Figure 5.25: Linearized iterations starting from the supersolution for BVP (5.15).

We observe from Figure5.26aand5.26bthat the solution starting from the subsolution increases monotonically and the solution starting from the supersolution decreases monotonically to the true solution.



(a) Monotonicity of solution starting from the subsolution.
(b) Monotonicity of solution starting from the supsolution.

Figure 5.26: Monotonicity of iterations for BVP (5.15).

Example 5.2.2.4 Consider the BVP on $\Omega = [0, 5]$

$$u^{''}(\xi) = -2u(\xi)\sin(u(\xi)). \tag{5.18}$$

The numerically calculated solutions are plotted in Figure5.27below.

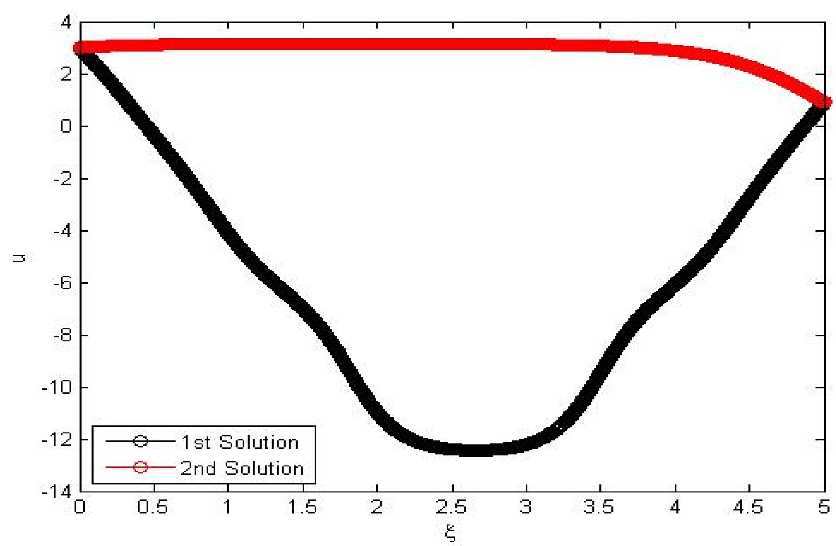


Figure 5.27: Numerically calculated exact solutions of BVP (5.18).

A subsolution of BVP (5.18) is

$$\underline{u}(\xi) = 2.8\xi^2 - 14.4\xi + 3, \tag{5.19}$$

and a supersolution is

$$\bar{u}(\xi) = \xi + 3, \tag{5.20}$$

Although this BVP does not satisfy property **[C2]**, numerically iterations (3.19) and (3.20) are still working. The numerical solutions of BVP (5.18) using the linearized iteration scheme (3.19) and (3.20) starting from subsolution and supersolution respectively are presented in Figure5.28and5.29.

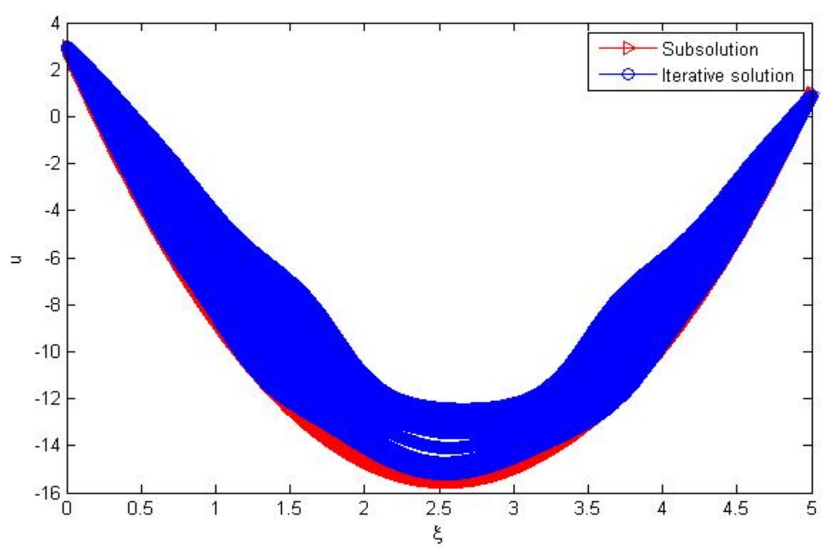


Figure 5.28: Linearized iterations starting from the subsolution for BVP (5.18).

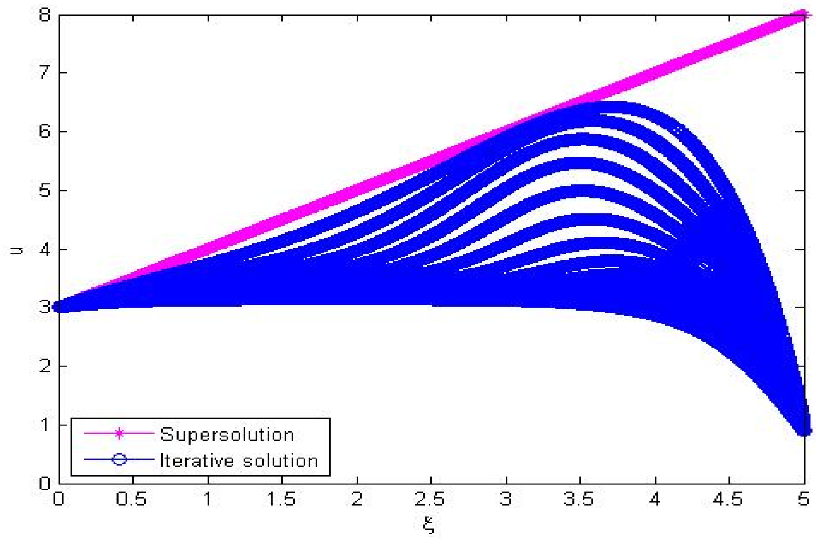
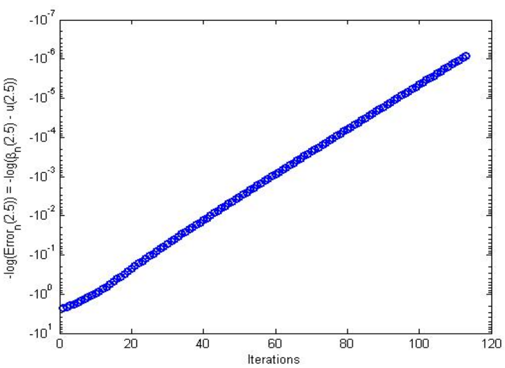
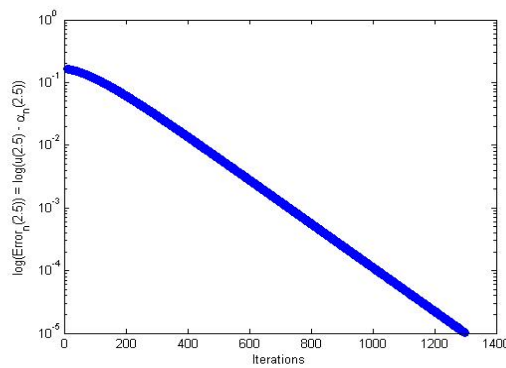


Figure 5.29: Linearized iterations starting from the supersolution for BVP (5.18).

Figure5.30aillustrates that iterations starting from subsolution are increasing monotonically, that is error tends to zero from a positive value. Likewise Figure5.30b shows that iterations starting from supersolution are decreasing monotonically that is error tends to zero from a negative value, when the error is calculated at a single point. Both of these iterations are converging to the true solution.



(a) Iterations starting from subsolution.



(b) Iterations starting from supersolution.

Figure 5.30: Monotonicity of the iterates for BVP (5.18).

5.3 Numerical Results for the linearized DD methods

5.3.1 Numerical results for the Linearized domain decomposition method to solve $u^{''} = f(\xi, u)$

To illustrate Theorem3.1.11we will provide some numerical experiments.

Example 5.3.1.1 Consider the BVP on $\Omega = [0, 1]$

$$u^{''}(\xi) = -20 + 60\pi\xi\cos(20\pi\xi^3) - \frac{(60\pi\xi^2)^2}{2}\sin(20\pi\xi^3), \qquad u = 0 \text{ on } \partial\Omega. \tag{5.21}$$

The exact solution of the BVP (5.21) is given by,

$$u(\xi) = 10\xi - 10\xi^2 + \frac{1}{2}\sin(20\pi\xi^3),$$

which is shown in Figure5.4. The numerical solution of BVP (5.21) obtained by Lui's linearized DD iterations (4.2) and (4.3) starting from the subsolution (5.3) and supersolution (5.4) are presented in Figures5.31and5.32respectively.

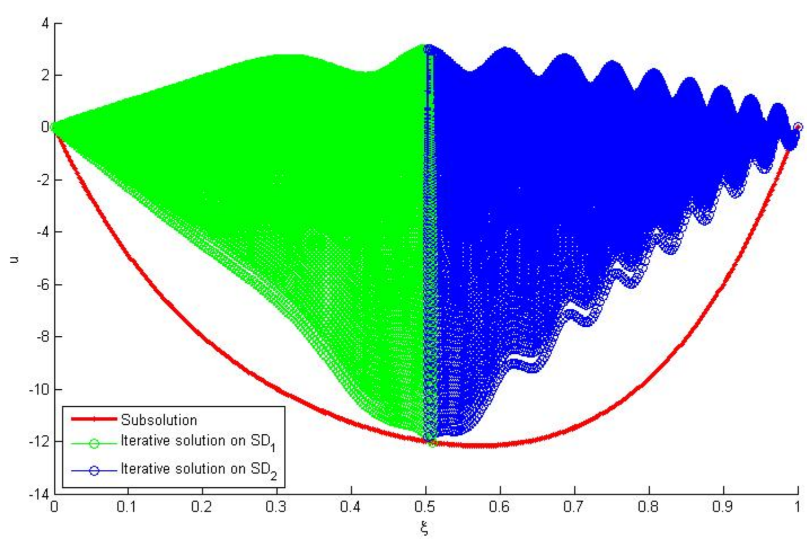


Figure 5.31: Linearized DD iterations starting from the subsolution for BVP (5.21).

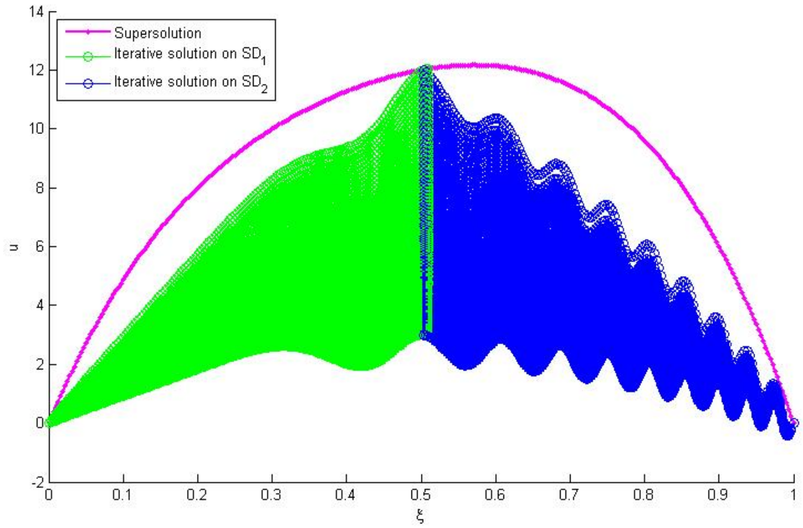
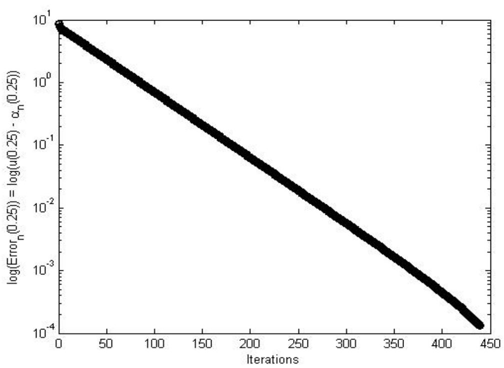
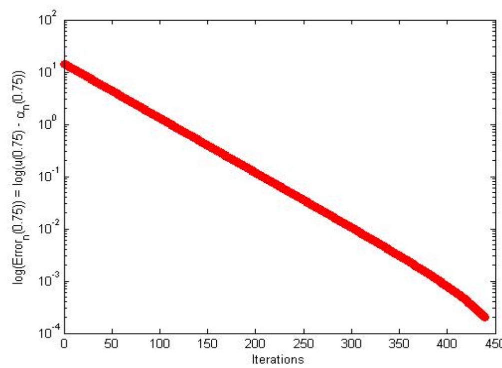


Figure 5.32: Linearized DD iterations starting from the supersolution for BVP (5.21).

We observe from Figure5.33that iterations starting from the subsolution increase monotonically on both of the subdomains.



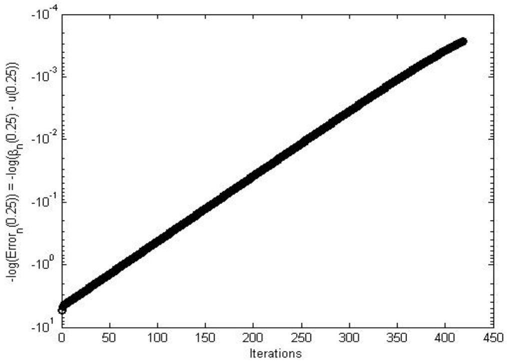
(a) Monotonicity of iteration on the first subdomain.



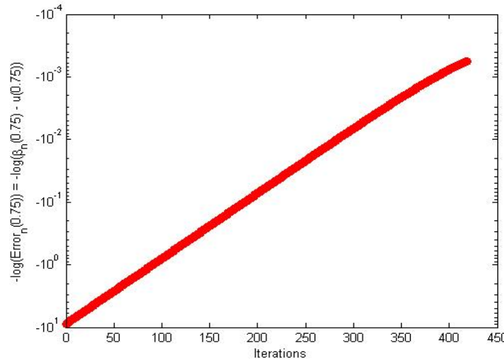
(b) Monotonicity of iteration on the second subdomain.

Figure 5.33: Monotonicity of iterates starting from subsolution for BVP (5.21).

Likewise iterations starting from the supersolution decrease monotonically on both of the subdomains, which is illustrated in Figure5.34.



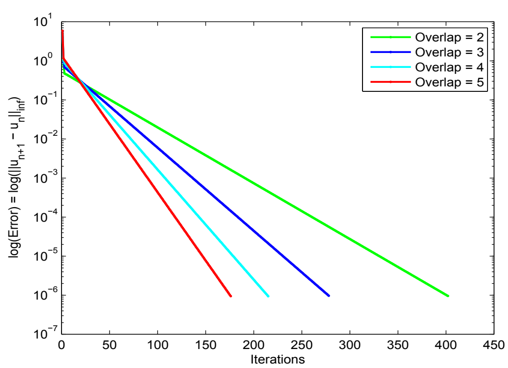
(a) Monotonicity of iteration on the first subdomain.



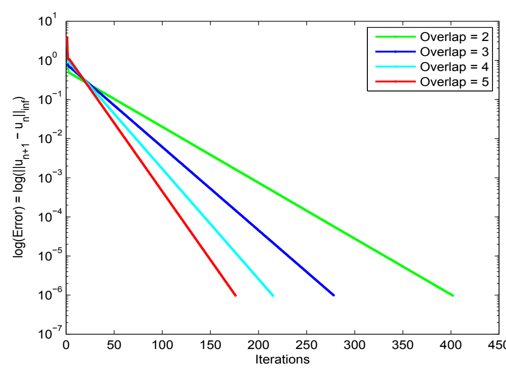
(b) Monotonicity of iteration on the second subdomain.

Figure 5.34: Monotonicity of iterates starting from supersolution for BVP (5.21).

Figure 5.35a and 5.35b indicate that as we increase the overlap the iterates converge more quickly. Whereas if we increase the number of subdomains the iterates



(a) Effect of the overlap on the 1st subdomain.



(b) Effect of the overlap on the 2nd subdomain.

Figure 5.35: Effect of the overlap for the linearized DD solution of BVP (5.21).

converge to the true solution more slowly as shown in Figure 5.36.

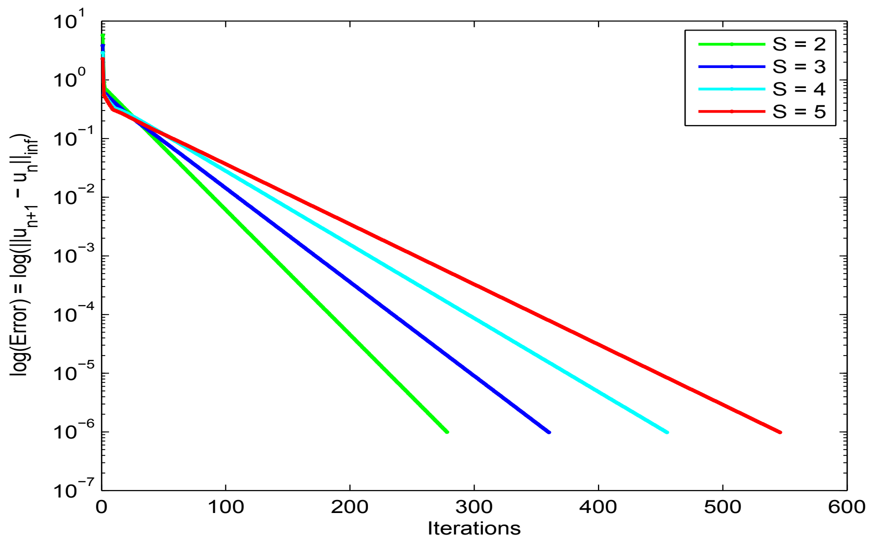


Figure 5.36: Effect of the number of subdomains on the linearized DD solution of BVP (5.21).

Figure5.37shows that the inequality $\underline{u} \leq u^{(n+\frac{1}{2})} \leq u^{(n+1)} \leq u^{(n+\frac{3}{2})} \leq \bar{u}$ is true when the iterates start from the subsolution. Likewise Figure5.38shows that the

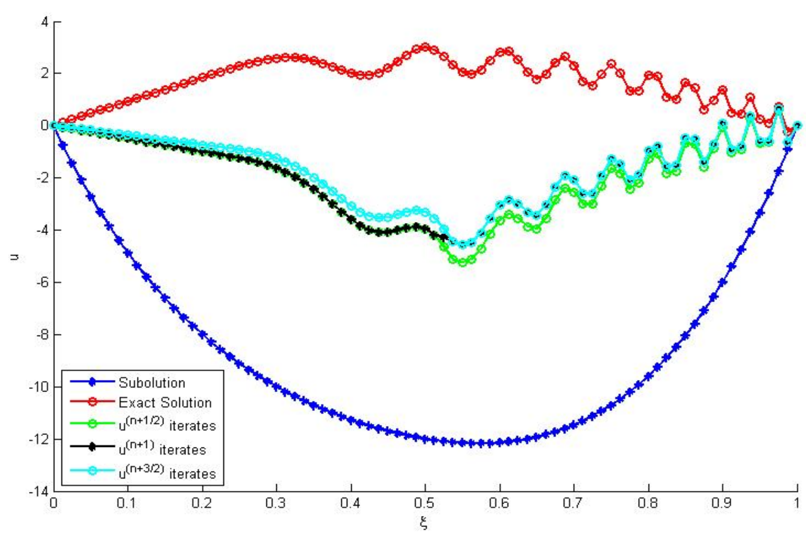


Figure 5.37: Relation between iterates (4.2) and (4.3) starting from subsolution of BVP (5.21) for $n = 9$.

similar inequality $\underline{u} \leq u^{(n+\frac{1}{2})} \leq u^{(n+1)} \leq u^{(n+\frac{3}{2})} \leq \bar{u}$ holds when the iterates start from the supersolution.

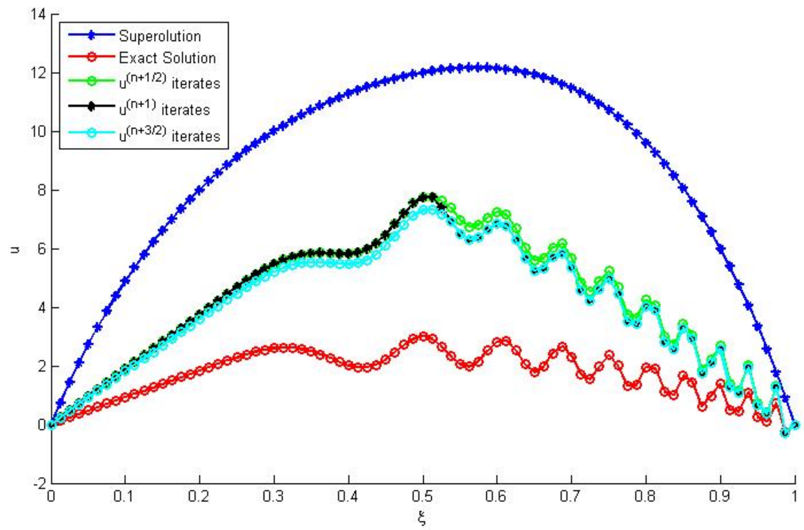


Figure 5.38: Relation between iterates (4.2) and (4.3) starting from supersolution of BVP (5.21) for $n = 9$.

Example 5.3.1.2 Consider the BVP,

$$-u^{''} = u - 3\xi - 5\tan(\xi), \qquad u(0) = 0 \qquad u(1) = 1. \tag{5.22}$$

The analytic solution of the BVP (5.22) is given by

$$u(\xi) = \frac{\sin(\xi)\{5\ln\left(\frac{1+\sin(1)}{\cos(1)}\right)\cos(1) - 2\}}{\sin(1)} + 3\xi - 5\ln\left\{\frac{1+\sin(\xi)}{\cos(\xi)}\right\}\cos(\xi).$$

The exact solution of equation (5.22) was shown in Figure5.8. Figures5.39and5.40 show that the numerical solution of BVP (5.22) using Lui's linearized DD iterations (4.1.1) starting from the subsolution (5.6) and the supersolution (5.7) respectively.

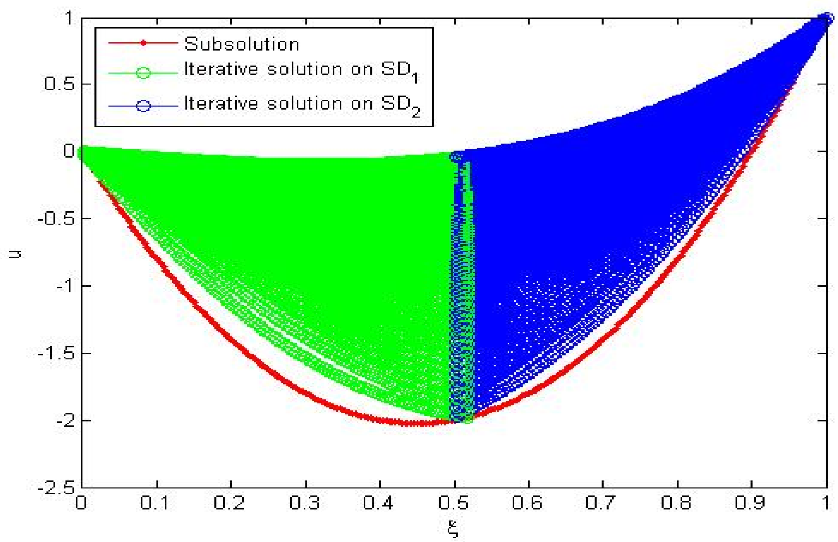


Figure 5.39: Linearized DD iterations starting from the subsolution for BVP (5.22).

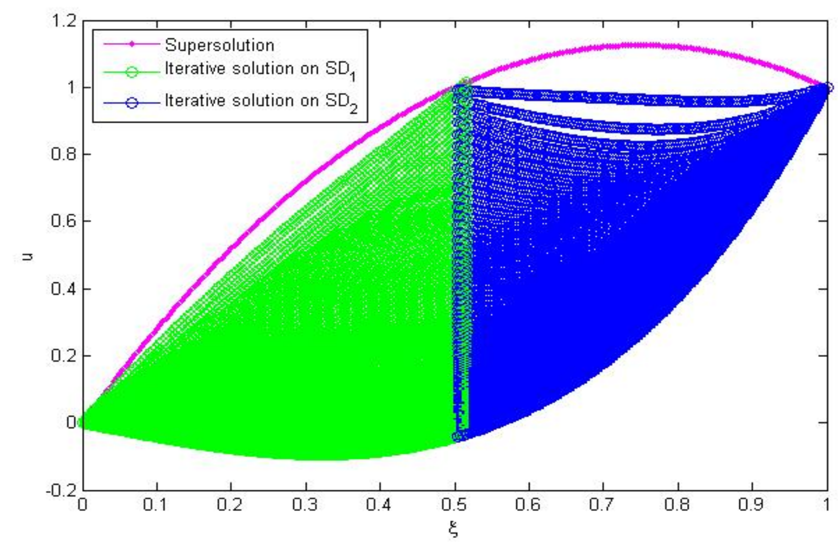
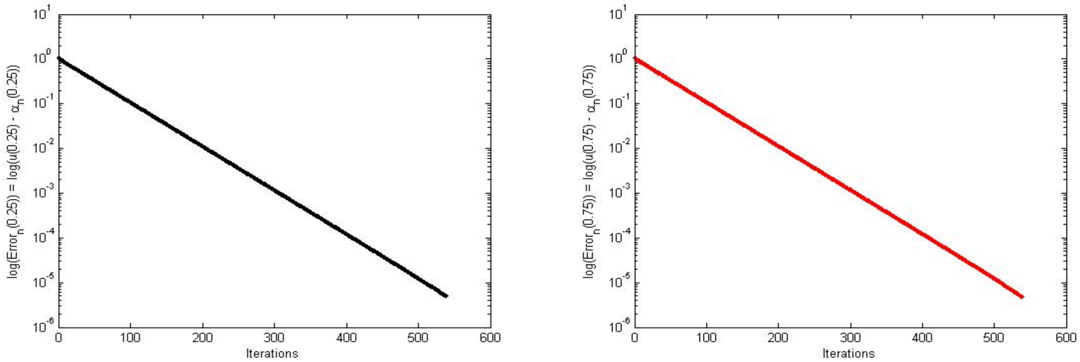


Figure 5.40: Linearized DD iterations starting from the supersolution for BVP (5.22)

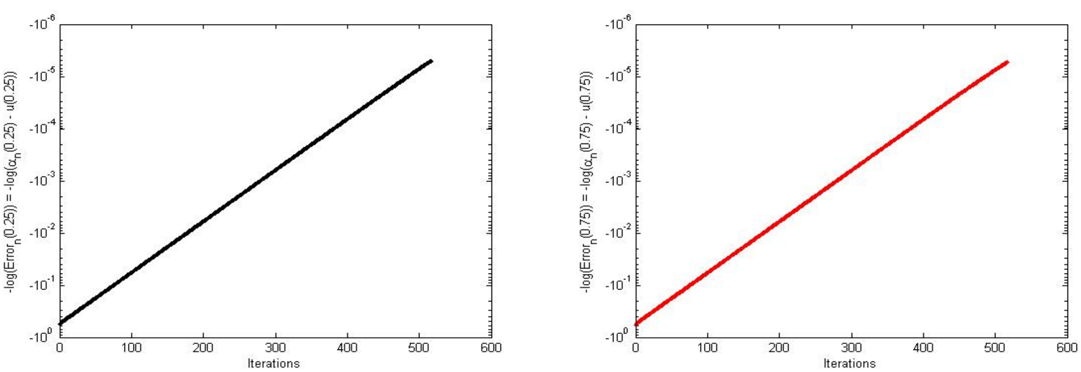
We observe from Figure5.41and5.42that solutions starting from the subsolution increase monotonically and solutions starting from the supersolution decrease monotonically and both these iterations converge to the true solution.



(a) Monotonicity of iterations on the first subdomain.

(b) Monotonicity of iterations on the second subdomain.

Figure 5.41: Monotonicity of the iterates starting from subsolution for BVP (5.22).

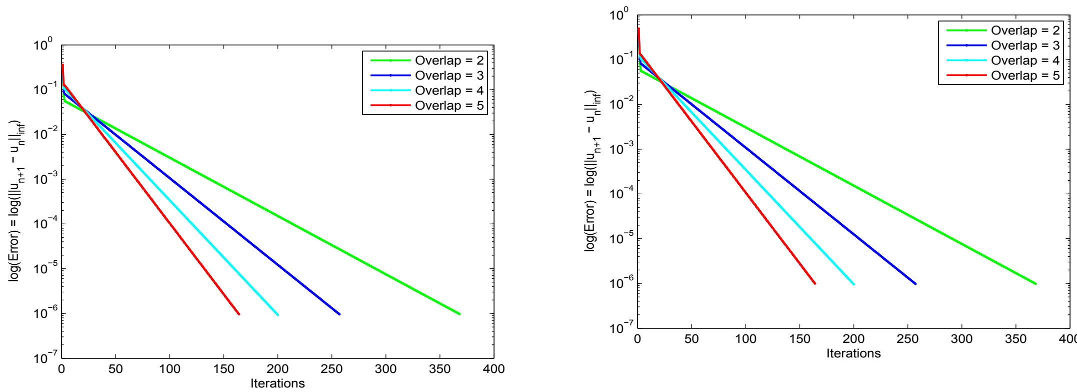


(a) Monotonicity of solution starting from supersolution on the first subdomain.

(b) Monotonicity of solution starting from supersolution on the second subdomain.

Figure 5.42: Monotonicity of the iterates starting from supersolution for BVP (5.22).

Figure5.43aand5.43bindicate that as we increase the overlap the linearized DD iterates converge more quickly.



(a) Effect of the overlap on the 1st subdomain.

(b) Effect of the overlap on the 2nd subdomain.

Figure 5.43: Effect of the overlap on the linearized DD solution of BVP (5.22).

Whereas if we increase the number of subdomains the linearized DD iterates converge to the true solution more slowly which is shown in Figure5.44.

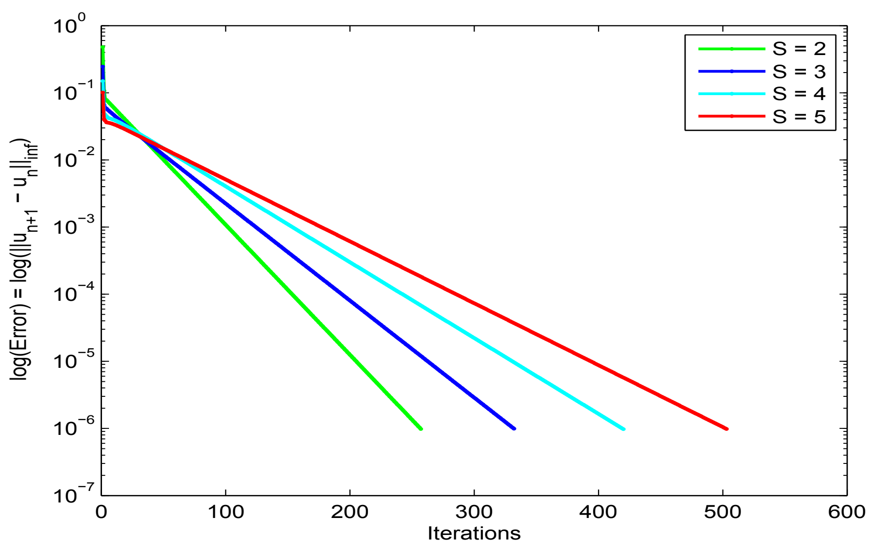


Figure 5.44: Effect of the number of subdomains on the linearized DD solution of BVP (5.22).

Figure5.45shows that inequality $\underline{u} \leq u^{(n+\frac{1}{2})} \leq u^{(n+1)} \leq u^{(n+\frac{3}{2})} \leq \bar{u}$ is true when the iterates start from the subsolution. Likewise Figure5.46shows that the

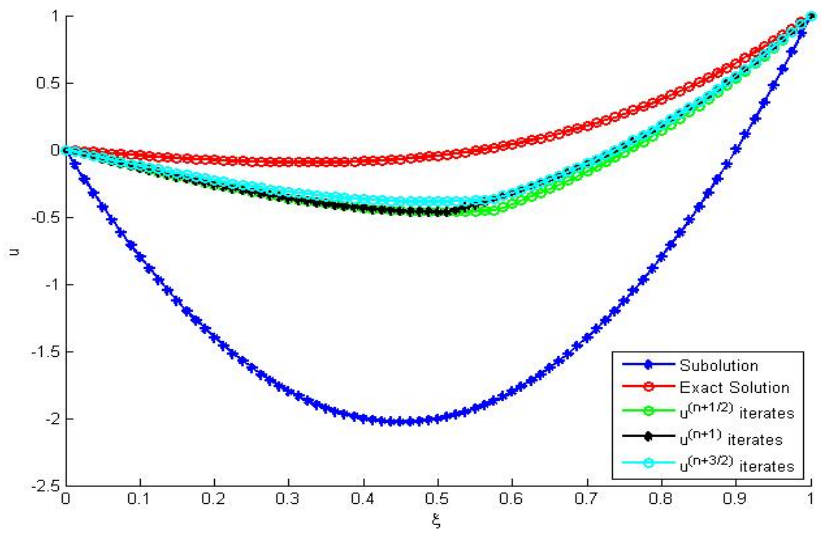


Figure 5.45: Relation between iterates (4.2) and (4.3) starting from subsolution of BVP (5.22) for $n = 9$.

similar inequality $\underline{u} \leq u^{(n+\frac{1}{2})} \leq u^{(n+1)} \leq u^{(n+\frac{3}{2})} \leq \bar{u}$ holds when the iterates start from supersolution.

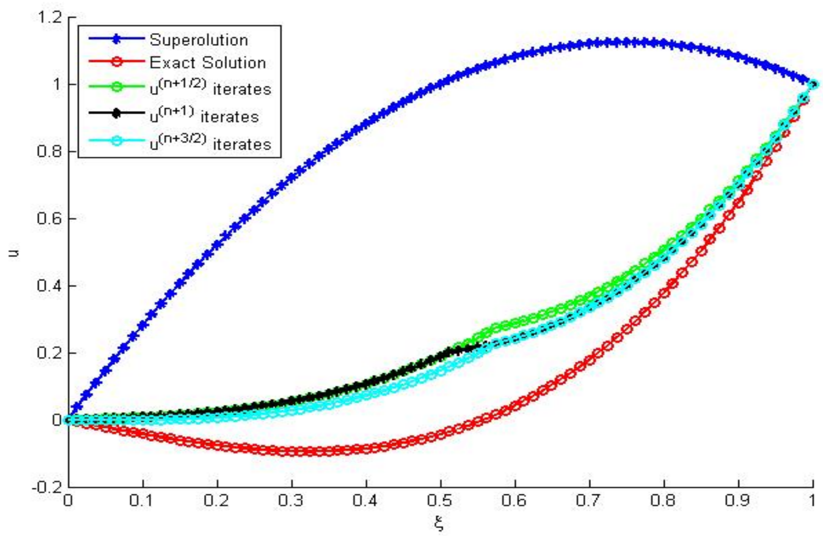


Figure 5.46: Relation between iterates (4.2) and (4.3) starting from supersolution of BVP (5.22) for $n = 9$.

Example 5.3.1.3 Consider the nonlinear BVP,

$$u^{''} = e^{u}, \qquad u(0) = 0 \qquad u(1) = 0. \tag{5.23}$$

The analytic solution of the BVP (5.23) is given by

$$u(\xi) = -\log(2) + 2\log[a\sec(\frac{a}{2}(\xi - \frac{1}{2}))],$$

The exact solution of equation (5.23) was shown in Figure5.12. Figures5.47and5.48 show the numerical solution of BVP (5.23) using Lui's linearized DD iterations (4.2) and (4.3) starting from the subsolution (5.9) and the supersolution (5.10) respectively.

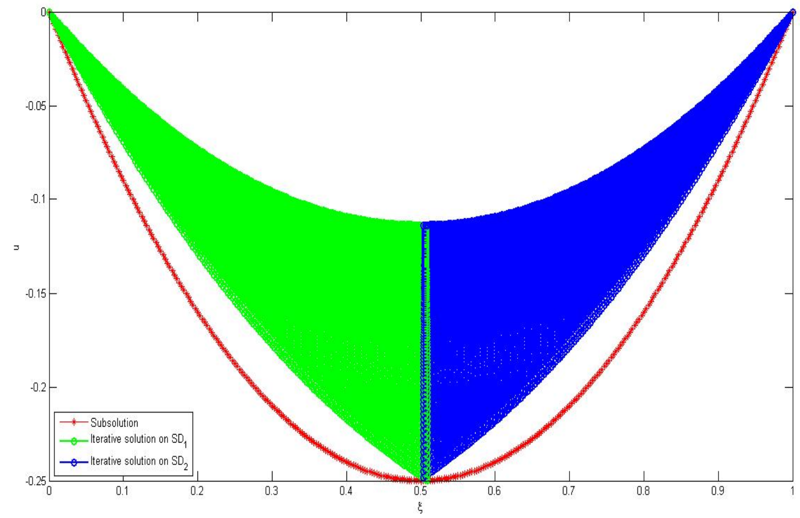


Figure 5.47: Linearized DD iterations starting from the subsolution for BVP (5.23).

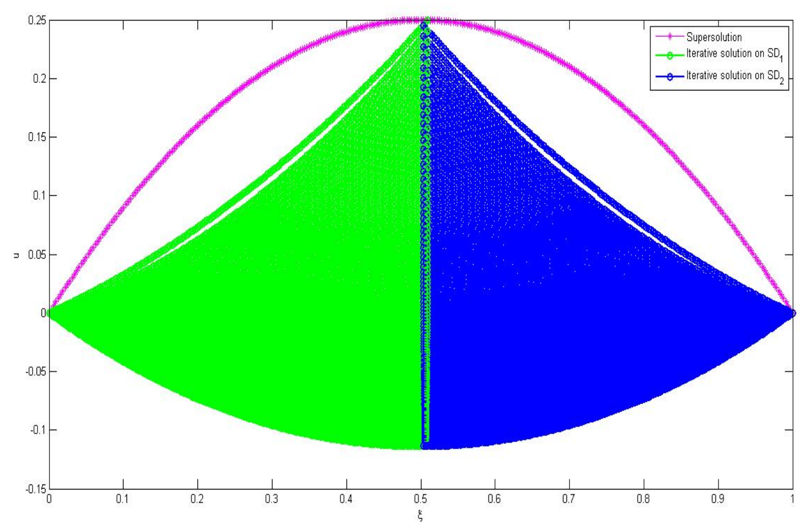
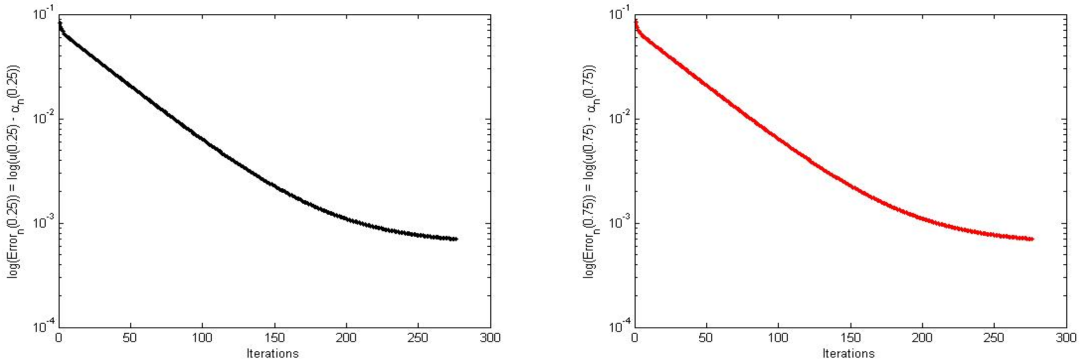


Figure 5.48: Linearized DD iterations starting from the supersolution for BVP (5.23).

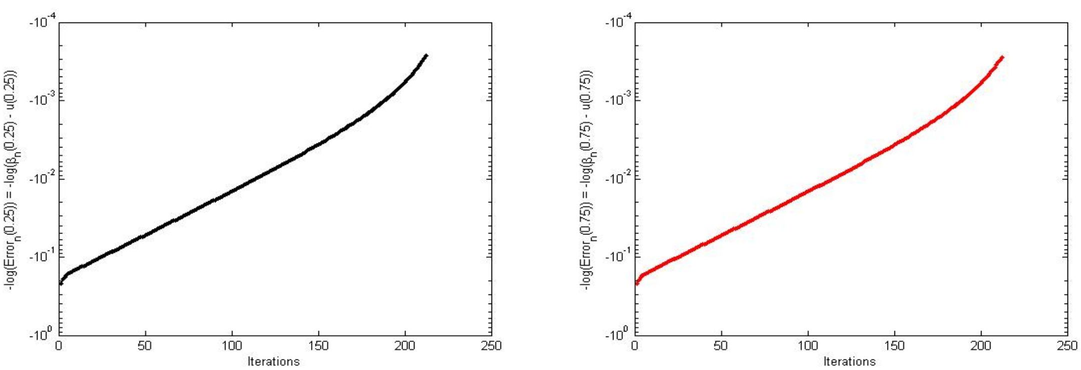
We observe from Figure5.49and5.50that solutions starting from the subsolution increases monotonically and solutions starting from the supersolution decreases monotonically and both these iterations converge to the true solution.



(a) Monotonicity of the iteration on the first subdomain.

(b) Monotonicity of the iteration on the second subdomain.

Figure 5.49: Monotonicity of the iterates starting from subsolution for BVP (5.23).

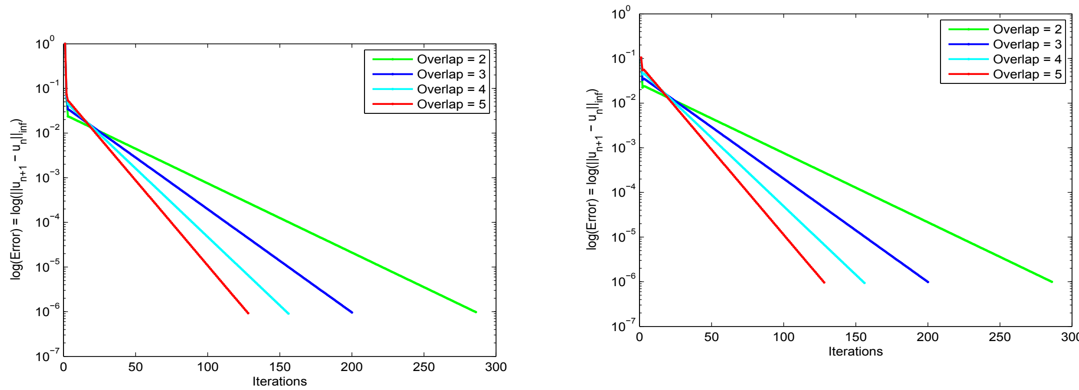


(a) Monotonicity of the iteration on the first subdomain.

(b) Monotonicity of the iteration on the second subdomain.

Figure 5.50: Monotonicity of the iterates starting from supersolution for BVP (5.23).

Figure5.51aand5.51bindicate that as we increase the overlap the linearized DD iterates converge more quickly.



(a) Effect of the overlap on the 1st subdomain.

(b) Effect of the overlap on the 2nd subdomain.

Figure 5.51: Effect of the overlap on the linearized DD solution of BVP (5.23).

Whereas if we increase the number of subdomains the linearized DD iterates converge to the true solution more slowly.

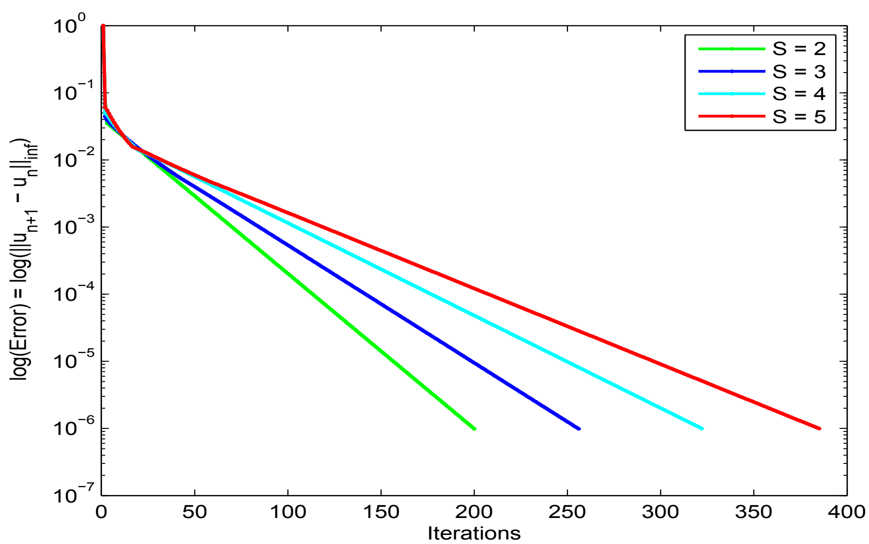


Figure 5.52: Effect of number of subdomains on the linearized DD solution of BVP (5.23).

Figure5.53shows that inequality $\underline{u} \leq u^{(n+\frac{1}{2})} \leq u^{(n+1)} \leq u^{(n+\frac{3}{2})} \leq \bar{u}$ is true when the iterates start from the subsolution.

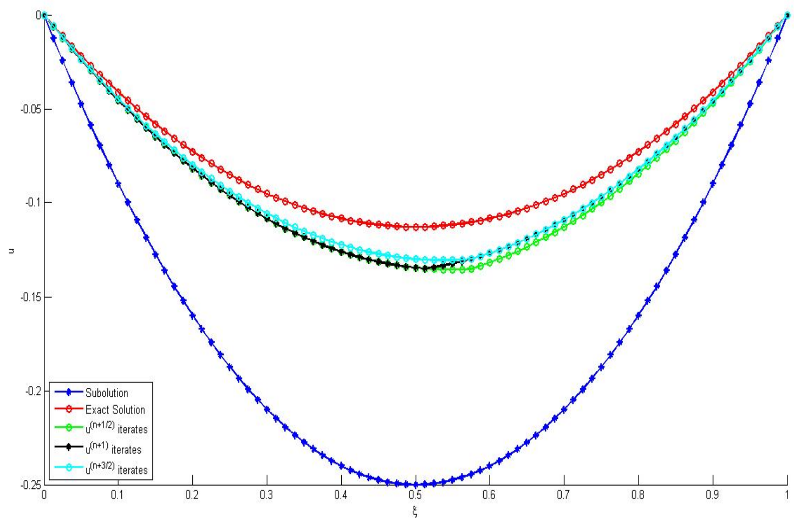


Figure 5.53: Relation between iterates (4.2) and (4.3) starting from subsolution of BVP (5.23) for $n = 9$.

Likewise Figure5.54shows that the similar inequality $\underline{u} \leq u^{(n+\frac{1}{2})} \leq u^{(n+1)} \leq u^{(n+\frac{3}{2})} \leq \bar{u}$ holds when the iterates start from the supersolution.

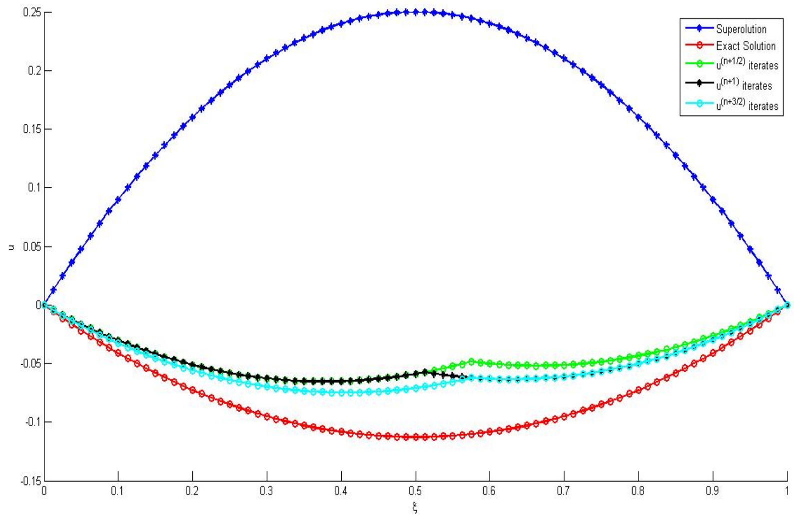


Figure 5.54: Relation between iterates (4.2) and (4.3) starting from supersolution of BVP (5.23) for $n = 9$.

5.3.2 Numerical results for the Linearized domain decomposition method to solve $u^{''} = f(\xi, u, u^{'})$

To illustrate Theorems4.2.5and4.2.6here we will provide some numerical experiments.

Example 5.3.2.1 Consider the BVP on $\Omega = [0, 1]$

$$u^{''}(\xi) = -20 + 60\pi\xi\cos(20\pi\xi^3) - \frac{(60\pi\xi^2)^2}{2}\sin(20\pi\xi^3), \qquad u = 0 \text{ on } \partial\Omega. \quad (5.24)$$

The analytic solution of the BVP (5.24) is given by,

$$u(\xi) = 10\xi - 10\xi^2 + \frac{1}{2}\sin(20\pi\xi^3),$$

which was plotted on Figure5.4. The numerical solution of BVP (5.24) we are getting from iterations (4.17) and (4.18) starting from the subsolution (5.3) is presented in Figure5.55.

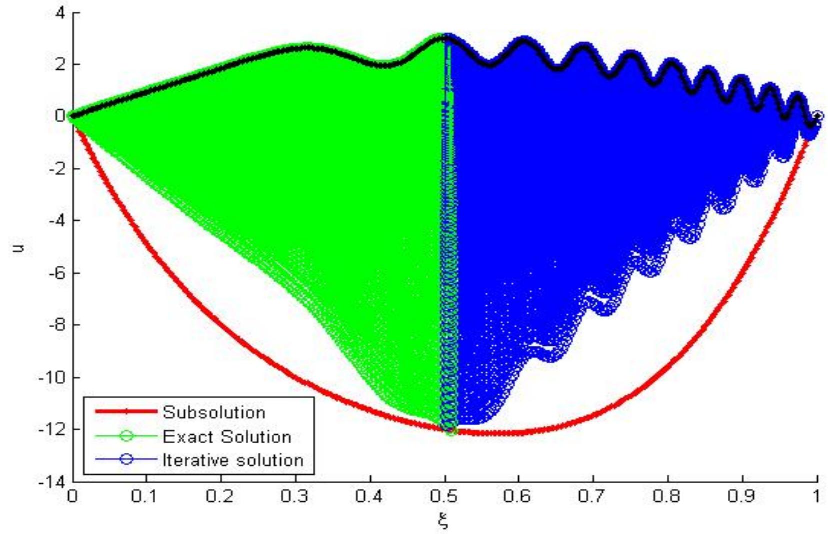


Figure 5.55: Linearized DD iterations starting from the subsolution for BVP (5.24).

The numerical solution of BVP (5.24) we are getting from iterations (4.27) and (4.28) starting from the supersolution (5.4) is presented in Figure5.56.

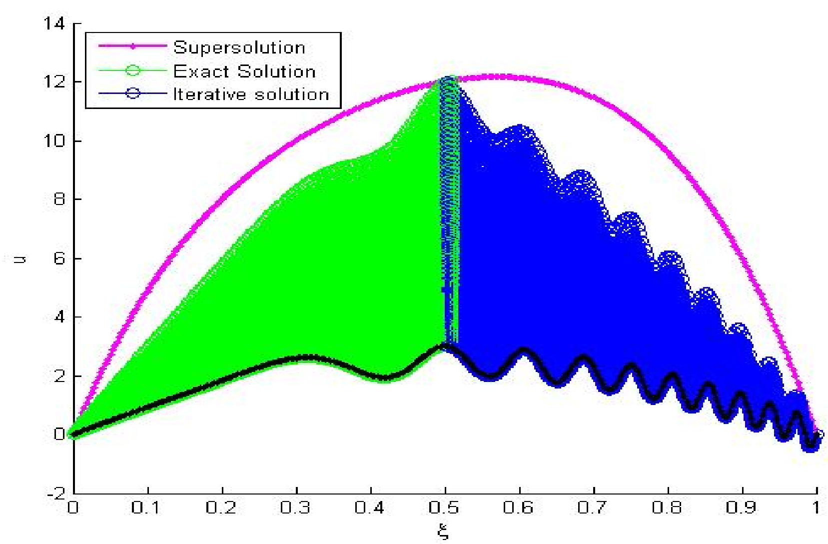
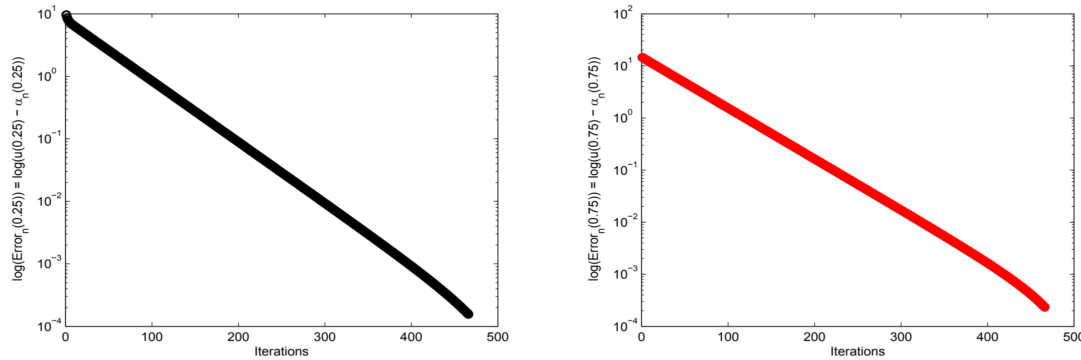


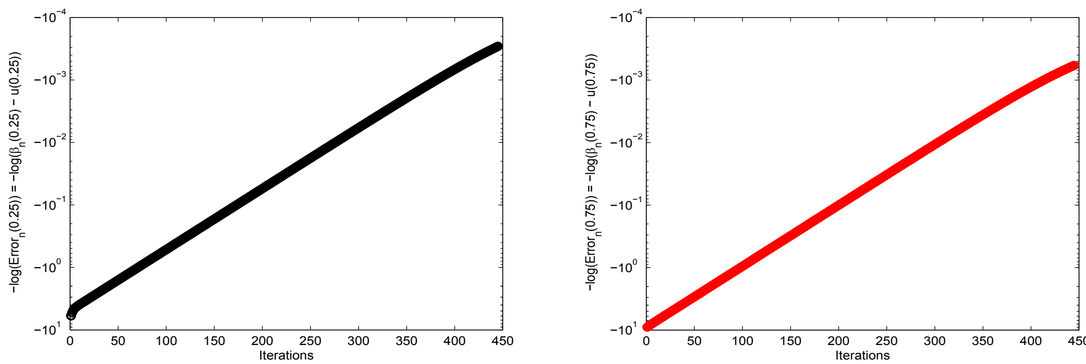
Figure 5.56: Linearized DD iterations starting from the supersolution for BVP (5.24).



(a) Monotonicity of iterations on the first subdomain.

(b) Monotonicity of iterations on the second subdomain.

Figure 5.57: Monotonicity of iterates starting from subsolution for BVP (5.24).

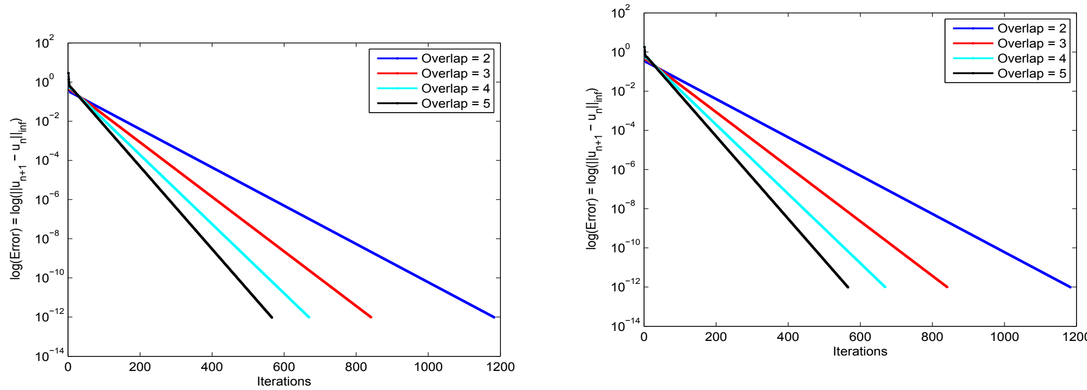


(a) Monotonicity of iterations on the first subdomain.

(b) Monotonicity of iterations on the second subdomain.

Figure 5.58: Monotonicity of iterates starting from supersolution for BVP (5.24).

We see from Figure5.57and5.58that the iterations starting from the subsolution increase monotonically and iterations starting from the supersolution decrease monotonically and both of these iterations converge to the true solution. Figure5.59a and5.59bindicate that as we increase the overlap the linearized DD iterates converge more quickly.



(a) Effect of the overlap on the 1st subdomain.

(b) Effect of the overlap on the 2nd subdomain.

Figure 5.59: Effect of the overlap on the linearized DD solution of BVP (5.24).

On the other hand if we increase the number of subdomains the iterates converge to the true solution more slowly.

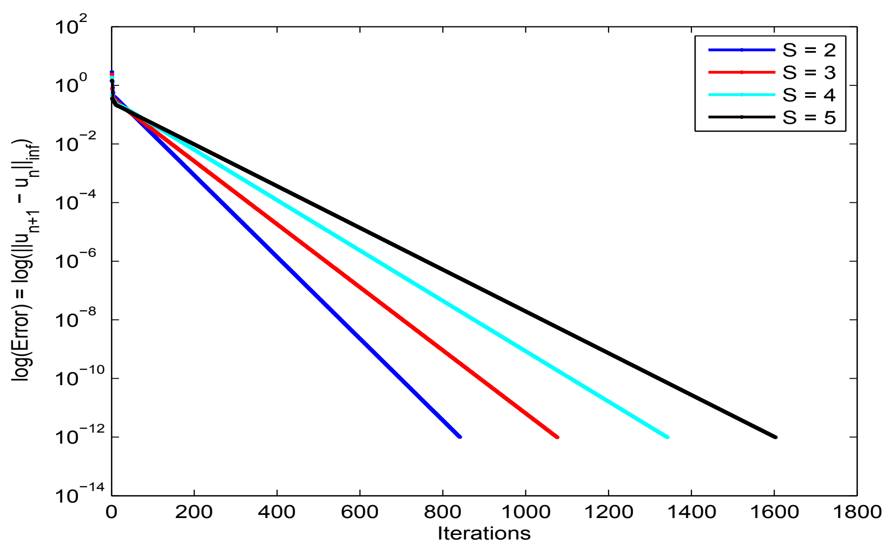


Figure 5.60: Effect of the number of subdomains on the linearized DD solution of BVP (5.24).

Figure5.61shows that inequality

$$\underline{\alpha} \leq \alpha_{(n)} \leq \alpha_{(n+\frac{1}{2})} \leq \alpha_{(n+1)} \leq \alpha_{(n+\frac{3}{2})} \leq \bar{\beta}, \tag{5.25}$$

is true for BVP (5.24), where the iterates start from the subsolution.

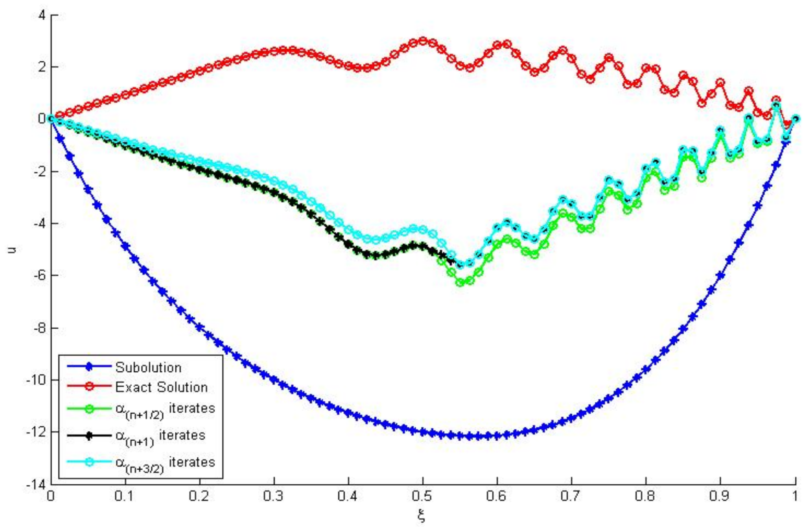


Figure 5.61: Plot showing inequality (5.25) for BVP (5.24) for $n = 9$.

Likewise Figure5.61shows that inequality

$$\bar{\beta} \geq \beta_{(n)} \geq \beta_{(n+\frac{1}{2})} \geq \beta_{(n+1)} \geq \beta_{(n+\frac{3}{2})} \geq \underline{\alpha}, \tag{5.26}$$

is true for BVP (5.24), where the iterates start from the supersolution.

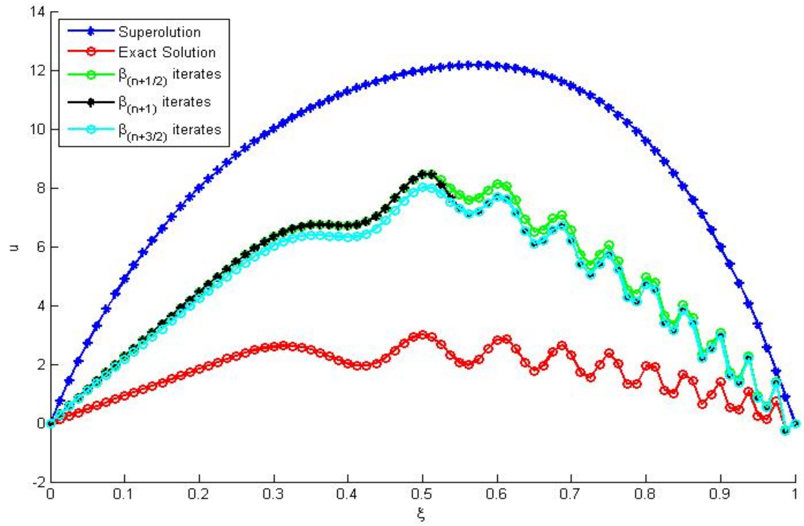


Figure 5.62: Plot showing inequality (5.26) for BVP (5.24) for $n = 9$.

Example 5.3.2.2 Consider the BVP

$$-u^{''} = u^{'} - 3x, \qquad u(0) = 0, \quad u(1) = 0. \tag{5.27}$$

whose analytic solution is given by

$$u(\xi) = \frac{3}{2}x^2 + \frac{3e^{-x}}{2(e^{-1}-1)} - 3x - \frac{3}{2(e^{-1}-1)}.$$

The exact solution of equation (5.27) was shown in Figure5.19. Figure5.63shows the numerical solution of BVP (5.27) we are getting from iterations (4.17) and (4.18) starting from the subsolution (5.13).

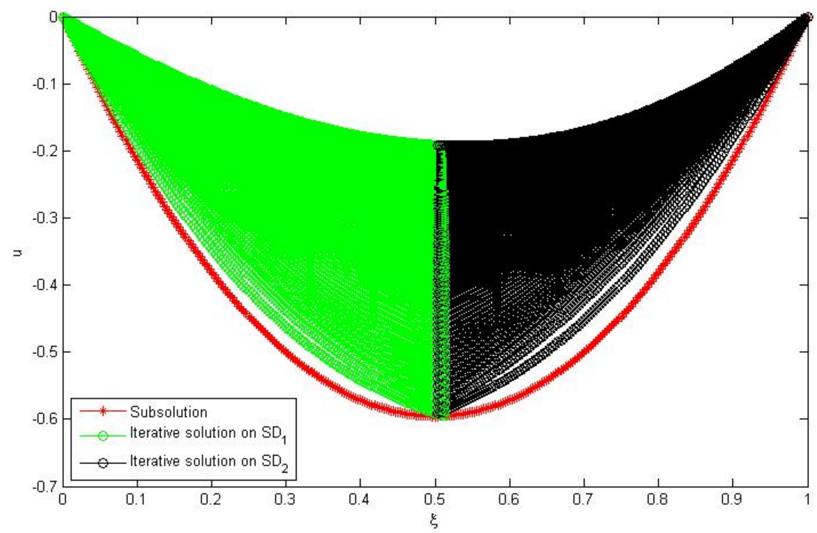


Figure 5.63: Linearized DD iterations starting from the subsolution for BVP (5.27).

Figure5.64shows the numerical solution of BVP5.27using iterations (4.27) and (4.28) starting from the supersolution (5.14).

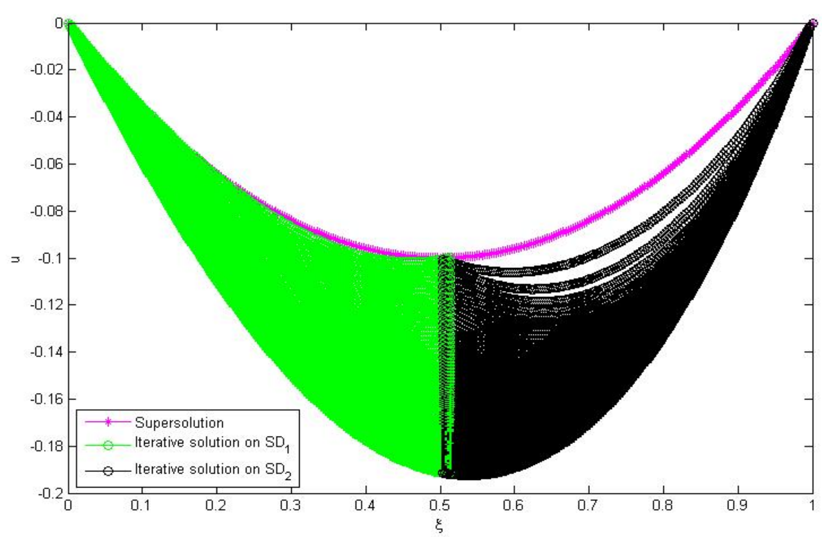
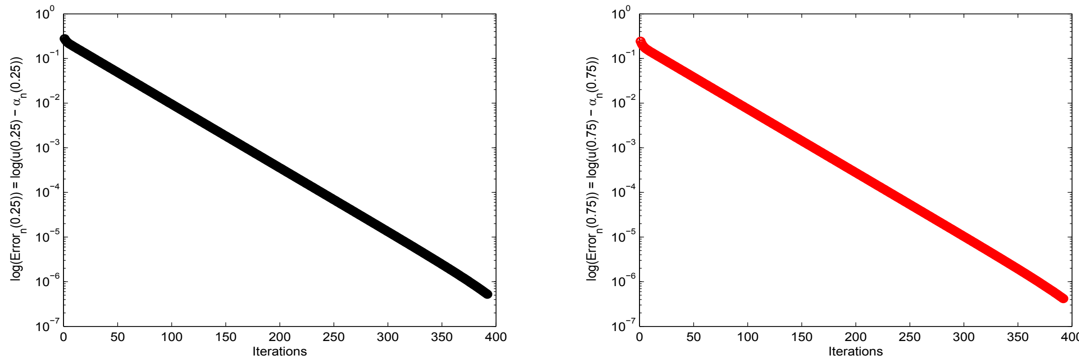


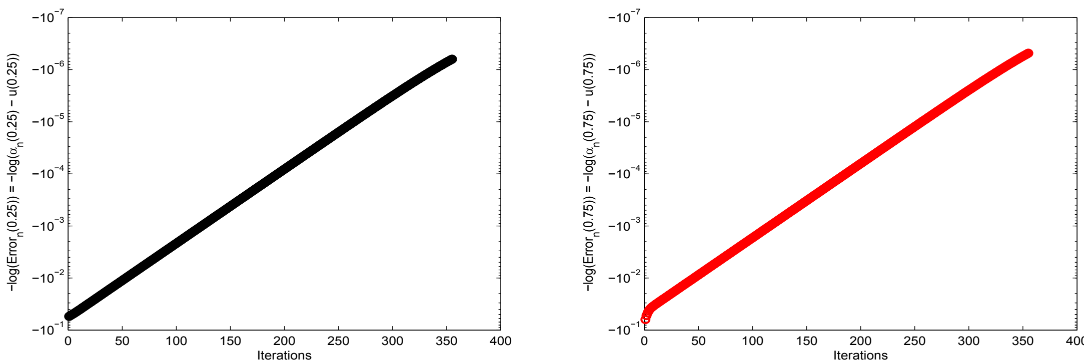
Figure 5.64: Linearized DD iterations starting from the supersolution for BVP (5.27).



(a) Monotonicity of iterations on the first subdomain.

(b) Monotonicity of iterations on the second subdomain.

Figure 5.65: Monotonicity of iterates starting from subsolution for BVP (5.27).

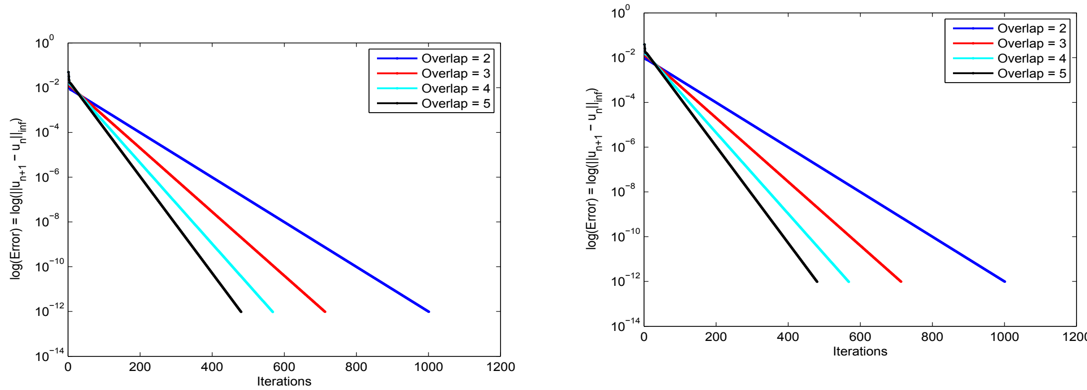


(a) Monotonicity of iterations on the first subdomain.

(b) Monotonicity of iterations on the second subdomain.

Figure 5.66: Monotonicity of iterates starting from supersolution for BVP (5.27).

We observe from Figure5.65and5.66that the solutions starting from the subsolutions increases monotonically and the solution starting from the supersolution decreases monotonically and both these iterations converge to the true solution. Figure5.67aand5.67bindicate that as we increase the overlap the linearized DD iterates converge more quickly.



(a) Effect of the overlap on the 1st subdomain.

(b) Effect of the overlap on the 2nd subdomain.

Figure 5.67: Effect of the overlap on the linearized DD solution of BVP (5.27).

Whereas if we increase the number of subdomains the iterates converge to the true solution more slowly.

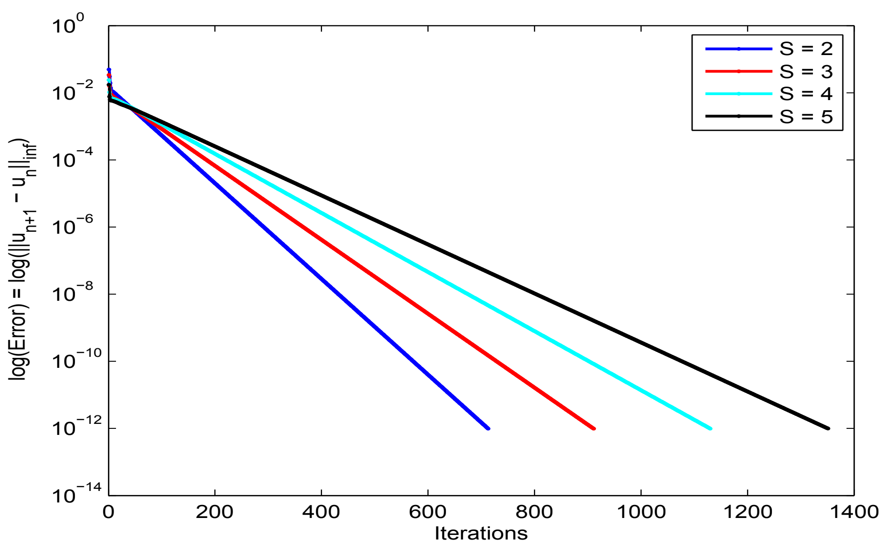


Figure 5.68: Effect of the number of subdomains on the linearized DD solution of BVP (5.27).

Figure5.69shows that inequality (5.25) is true for BVP (5.27), where the iterates start from the subsolution.

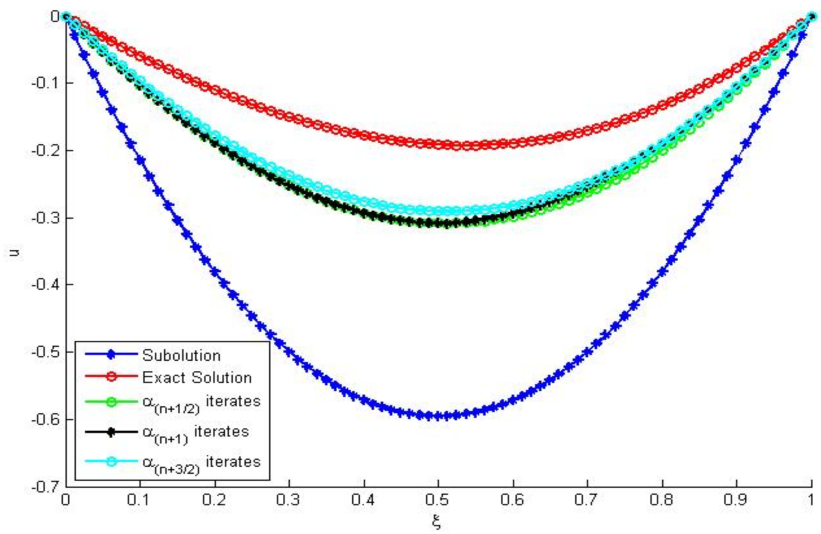


Figure 5.69: Plot showing inequality (5.25) for BVP (5.27) for $n = 9$.

Likewise Figure5.70shows that inequality (5.26) is true for BVP (5.24), where the iterates start from the supersolution.

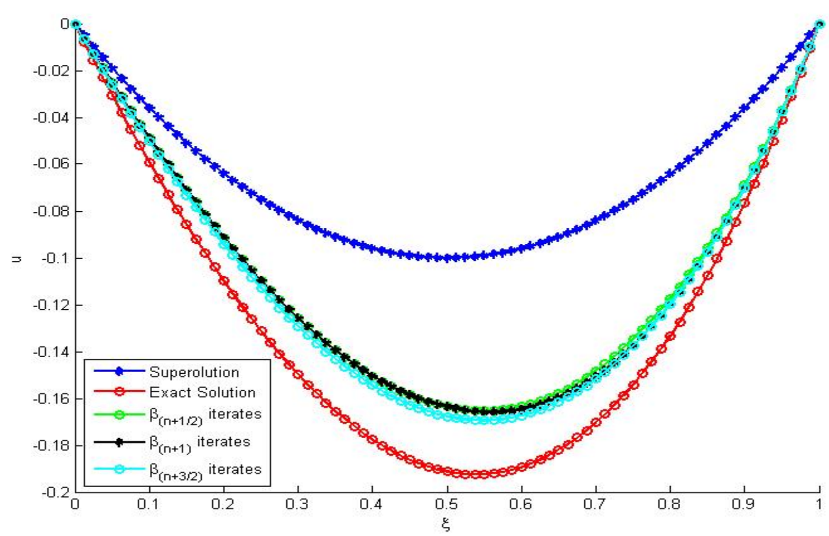


Figure 5.70: Plot showing inequality (5.26) for BVP (5.27) for $n = 9$.

Example 5.3.2.3 Consider the nonlinear BVP

$$u'' = \sin(u') + 2 - \sin(2\xi - 1), \qquad u(0) = 0, \quad u(1) = 0, \tag{5.28}$$

whose exact solution is given by

$$u(\xi) = \xi^2 - \xi.$$

The exact solution of equation (5.15) is shown in Figure5.23. Figure5.71shows the numerical solution of BVP (5.28) using iterations (4.17) and (4.18) starting from the subsolution (5.16).

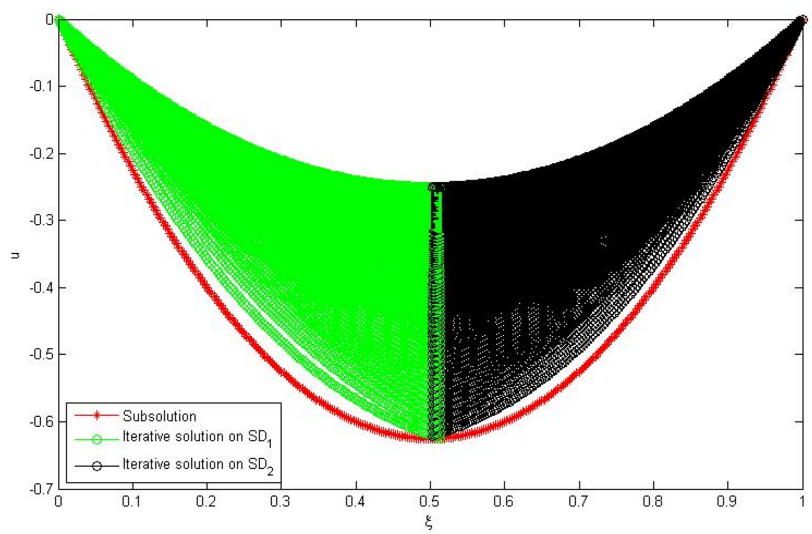


Figure 5.71: Linearized DD iterations starting from the subsolution for BVP (5.15).

Figure5.72shows the numerical solution of BVP (5.28) using iterations (4.27) and (4.28) starting from the supersolution (5.17).

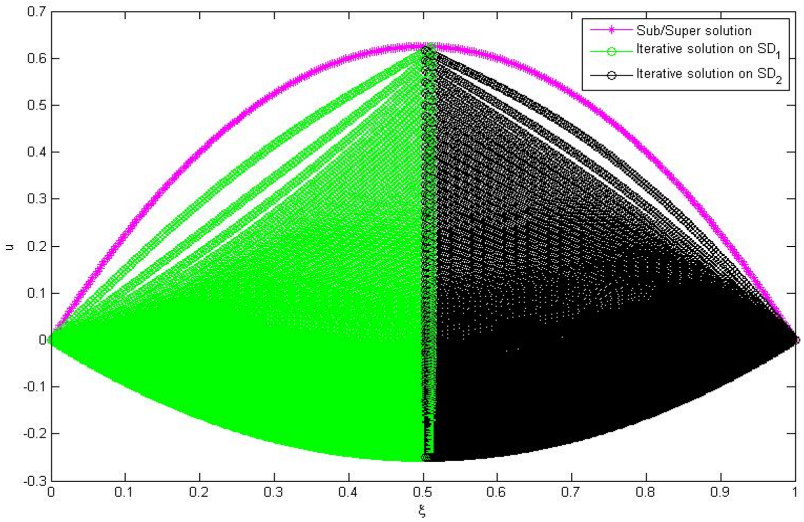
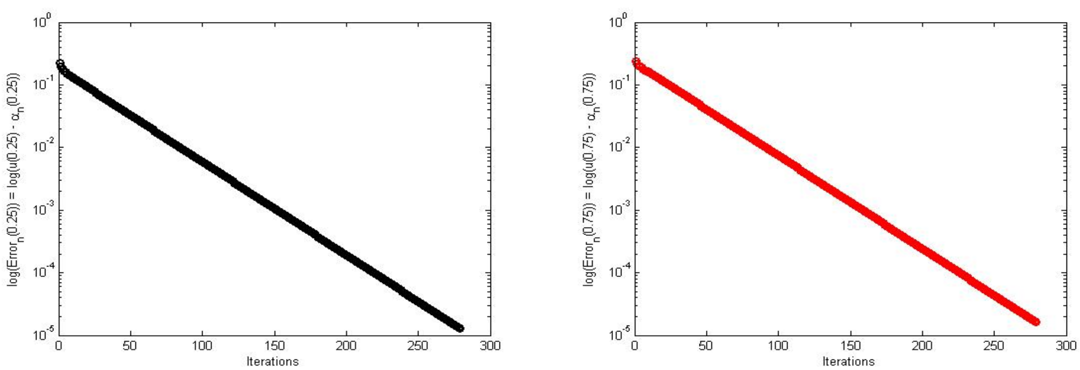


Figure 5.72: Linearized DD iterations starting from the supersolution for BVP (5.15).

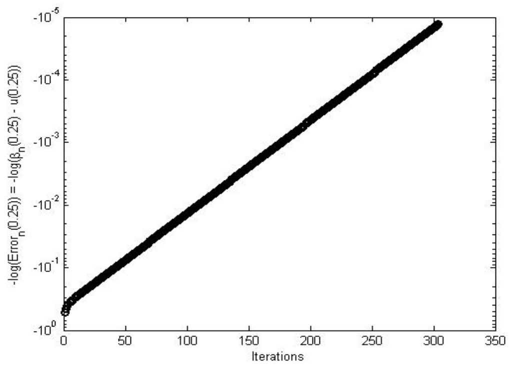
We observe from Figure5.73and5.74that the solutions starting from the subsolution increase monotonically and the solution starting from the supersolution decrease monotonically to the true solution.

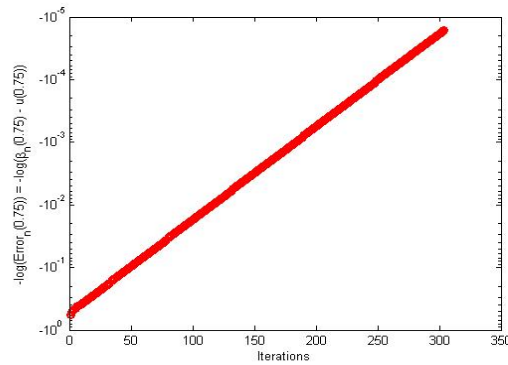


(a) Monotonicity of iterates on the first subdomain.

(b) Monotonicity of iterates on the second subdomain.

Figure 5.73: Monotonicity of iterates starting from subsolution for BVP (5.15).





(a) Monotonicity of iterations on the first subdomain.

(b) Monotonicity of iterations on second subdomain.

Figure 5.74: Monotonicity of iterates starting from supersolution for BVP (5.15).

Figure5.75shows that inequality (5.25) is true for BVP (5.15), where the iterates start from the subsolution.

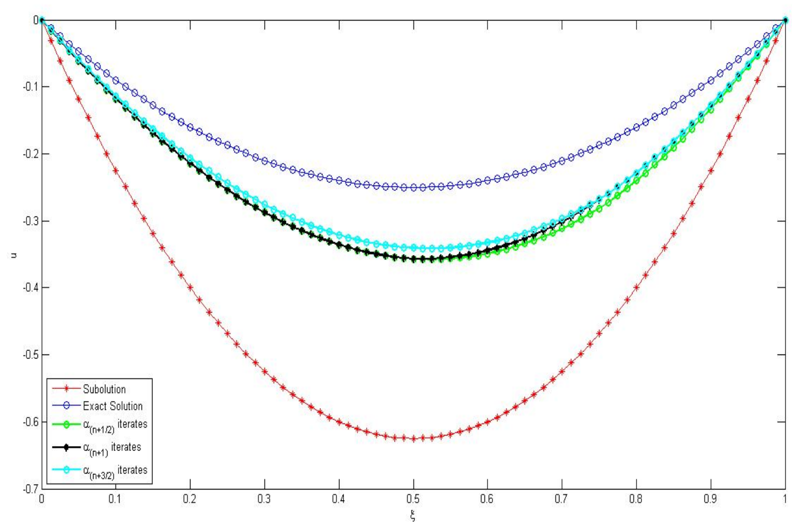


Figure 5.75: Plot showing inequality (5.25) for BVP (5.15) for $n = 9$.

Similarly Figure5.76shows that inequality (5.26) is true for BVP (5.15), where the iterates start from the supersolution.

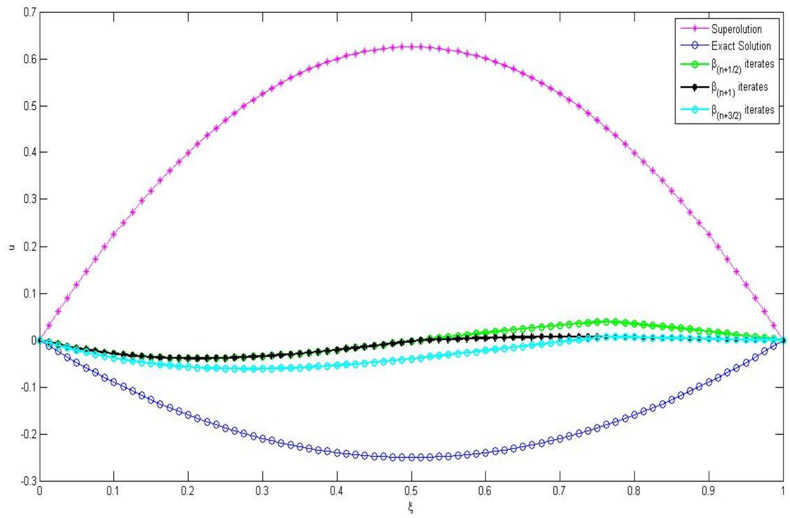


Figure 5.76: Plot showing inequality (5.25) for BVP (5.15) for $n = 9$.

Example 5.3.2.4 Consider the BVP on $\Omega = [0, 5]$

$$u^{''}(\xi) = -2u(\xi)\sin(u(\xi)). \tag{5.29}$$

The numerically calculated solutions of the BVP (5.29) were plotted in Figure5.27.

The numerical solution of BVP (5.29) obtained from iterations (4.17) and (4.18) starting from the subsolution (5.19) is presented in Figure5.77.

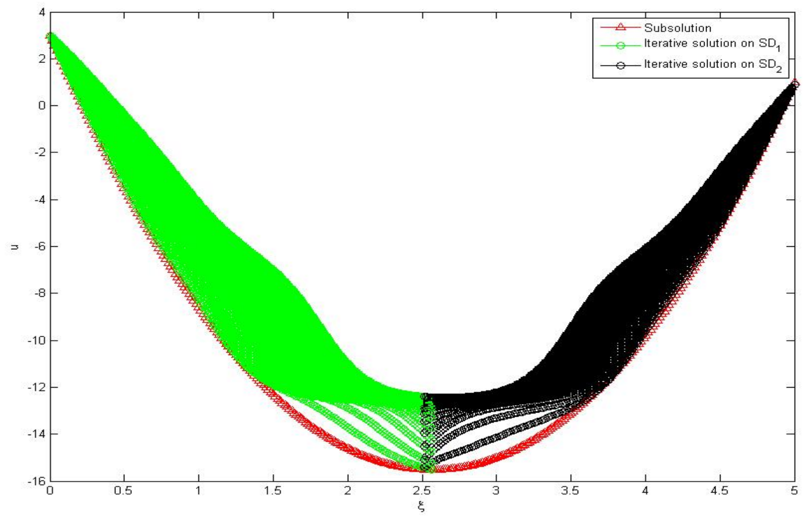


Figure 5.77: Linearized DD iterations starting from the subsolution for BVP (5.29).

The numerical solution of BVP (5.29) obtained from iterations (4.27) and (4.28) starting from the supersolution (5.20) is presented in Figure5.78.

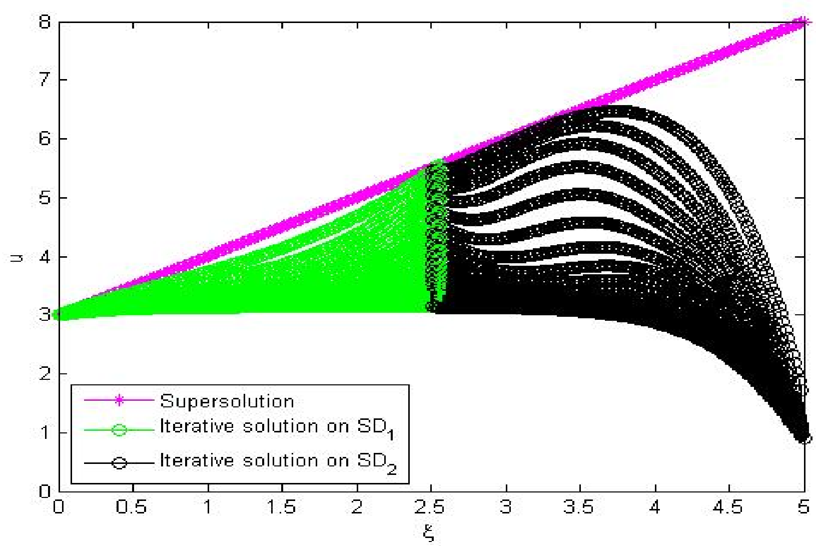
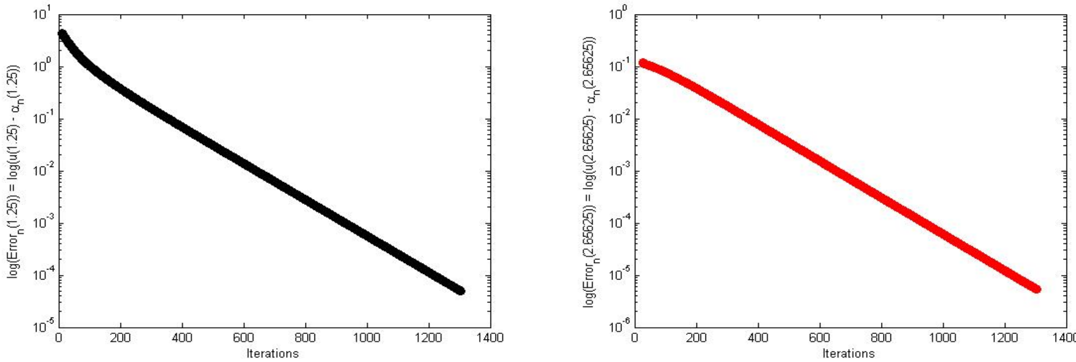


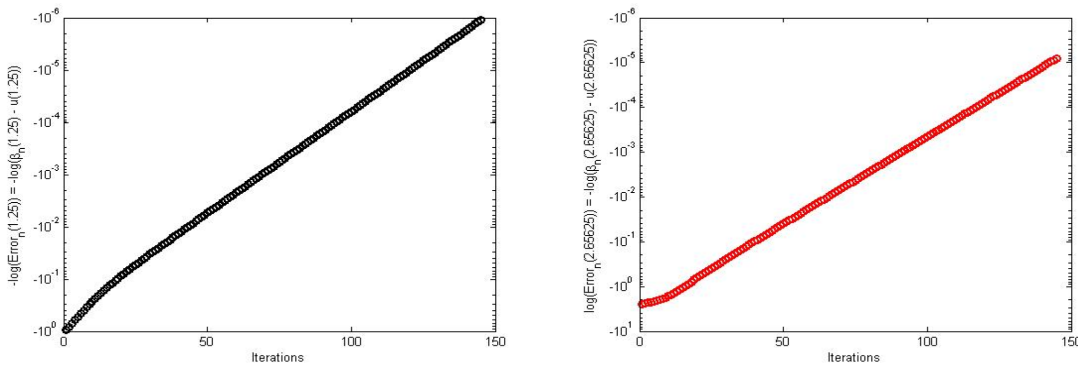
Figure 5.78: Linearized DD iterations starting from the supersolution for BVP (5.24).



(a) Monotonicity of iterations on the first subdomain.

(b) Monotonicity of iterations on the second subdomain.

Figure 5.79: Monotonicity of iterates starting from subsolution for BVP (5.29).

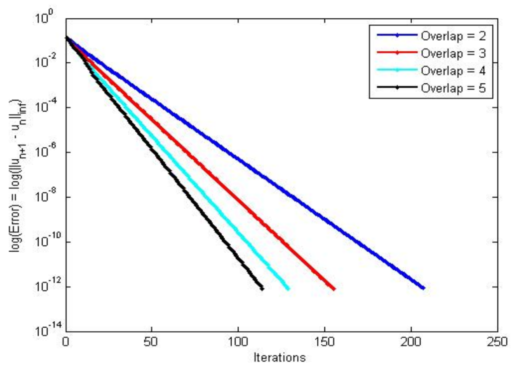


(a) Monotonicity of iterations on the first subdomain.

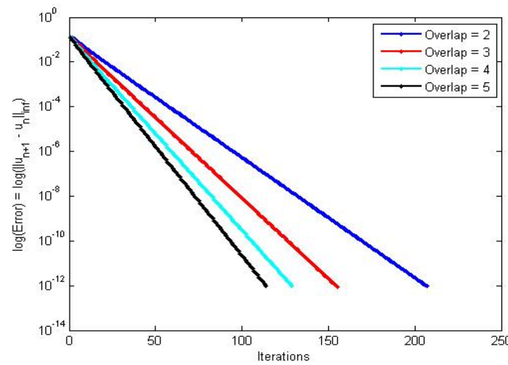
(b) Monotonicity of iterations on the second subdomain.

Figure 5.80: Monotonicity of iterates starting from supersolution for BVP (5.29).

We see from Figure5.79and5.80that the iterations starting from the subsolution increase monotonically and iterations starting from the supersolution decrease monotonically and both of these iterations converge to the true solution. Figure5.81a and5.81bindicate that as we increase the overlap the linearized DD iterates converge more quickly.



(a) Effect of the overlap on the 1st subdomain.



(b) Effect of the overlap on the 2nd subdomain.

Figure 5.81: Effect of the overlap on the linearized DD solution of BVP (5.29).

On the other hand if we increase the number of subdomains the iterates converge to the true solution more slowly.

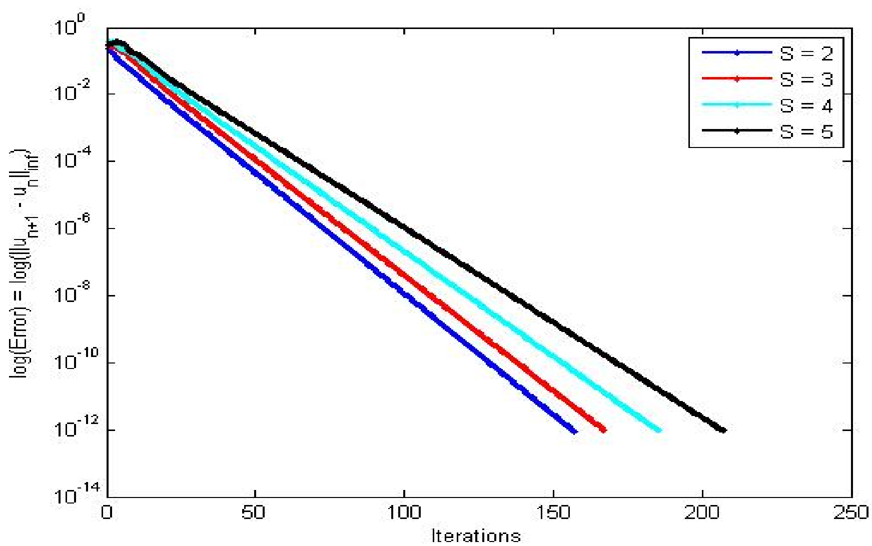


Figure 5.82: Effect of the number of subdomains on the linearized DD solution of BVP (5.24).

Figure5.83shows that the inequality

$$\underline{\alpha} \leq \alpha_{(n)} \leq \alpha_{(n+\frac{1}{2})} \leq \alpha_{(n+1)} \leq \alpha_{(n+\frac{3}{2})} \leq \bar{\beta}, \tag{5.30}$$

is true for BVP (5.29), when the iterates start from the subsolution.

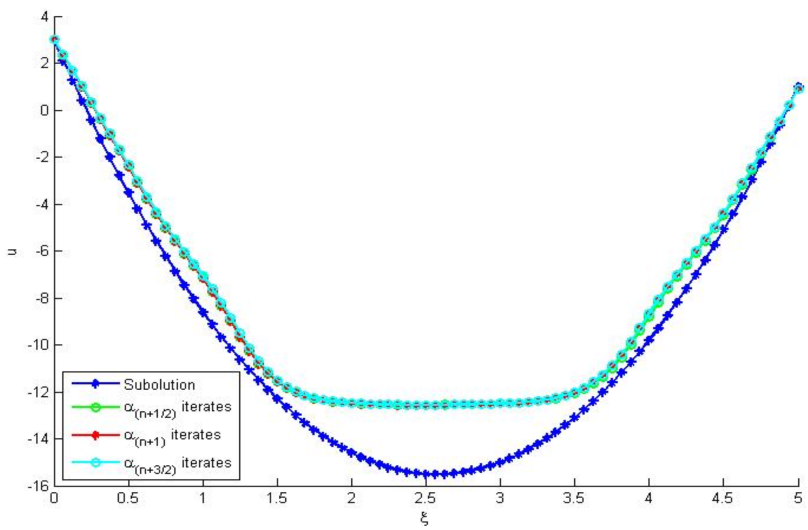


Figure 5.83: Plot showing inequality (5.25) for BVP (5.24) for $n = 9$.

Likewise Figure5.83shows that the inequality

$$\bar{\beta} \geq \beta_{(n)} \geq \beta_{(n+\frac{1}{2})} \geq \beta_{(n+1)} \geq \beta_{(n+\frac{3}{2})} \geq \underline{\alpha}, \tag{5.31}$$

is true for BVP (5.29), when the iterates start from the supersolution.

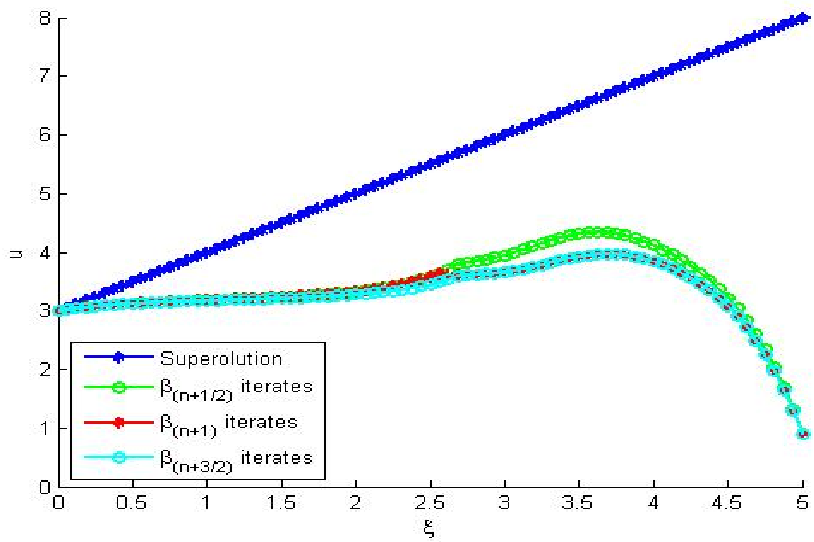


Figure 5.84: Plot showing inequality (5.31) for BVP (5.29) for $n = 9$.

5.4 Numerical result of moving mesh BVP

The BVP for the equidistribution principle is

$$(M(u)u^{'})^{'} = 0, \quad u(0) = 0, u(1) = 1.$$

Here we will use $M(u) = 1 + \gamma_1 \exp^{\frac{u-u_0}{\epsilon_1}} + \gamma_2 \exp^{\frac{u-u_N}{\epsilon_2}}$ with $\gamma_1 = 1$, $\gamma_2 = 1$, $\epsilon_1 = 1$ and $\epsilon_2 = 1$. We can rewrite above BVP as

$$u^{''} = -\frac{M^{'}(u)u^{'}}{M(u)}, \quad u(0) = 0, u(1) = 1. \tag{5.32}$$

Let us assume $v(\xi) = u(\xi) - \xi$, then at $\xi = 0, u = 0, v = 0$ and at $\xi = 1, u = 1, v = 0$. So the BVP (5.32) becomes,

$$v^{''} = -\frac{M^{'}(v+\xi)}{M(v+\xi)}(v^{'}+1)^2, \quad v(0) = 0, v(1) = 0. \tag{5.33}$$

This is of the form $v^{''} = f(\xi, v, v^{'})$ where $f(\xi, v, v^{'}) = -\frac{M^{'}(v+\xi)}{M(v+\xi)}(v^{'}+1)^2$. This $f(\xi, v, v^{'})$ unfortunately does not satisfy the global Lipschitz condition in $v^{'}$. For this BVP a subsolution is $v = \frac{1}{2}\xi^2 - \frac{1}{2}\xi$ and a supersolution is $v = C - \xi$ with $C \geq 1$. Although function $f(\xi, v, v')$ does not satisfy the conditions to solve the BVP (5.32) using iterations (3.19) and (3.20), numerically it seems like these iterations will work for this BVP. Further discussion about this is in the future work and conclusion Chapter..

Figure5.85and5.86show that single domain solutions generated by iterations (3.19) starting from the subsolution increase monotonically and solutions generated by iteration (3.20) starting from the supersolution decrease monotonically. Both of these iteration are converging to true solution.

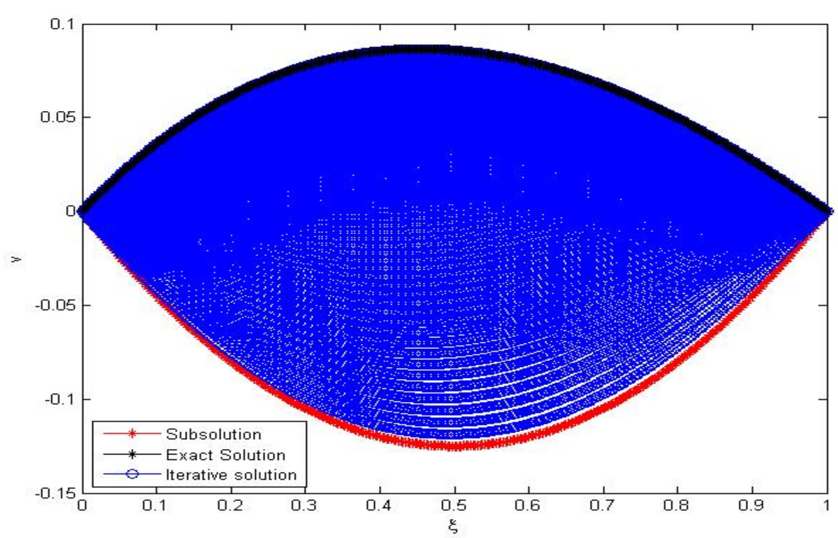


Figure 5.85: Iterations starting from the subsolution for the mesh BVP (5.32).

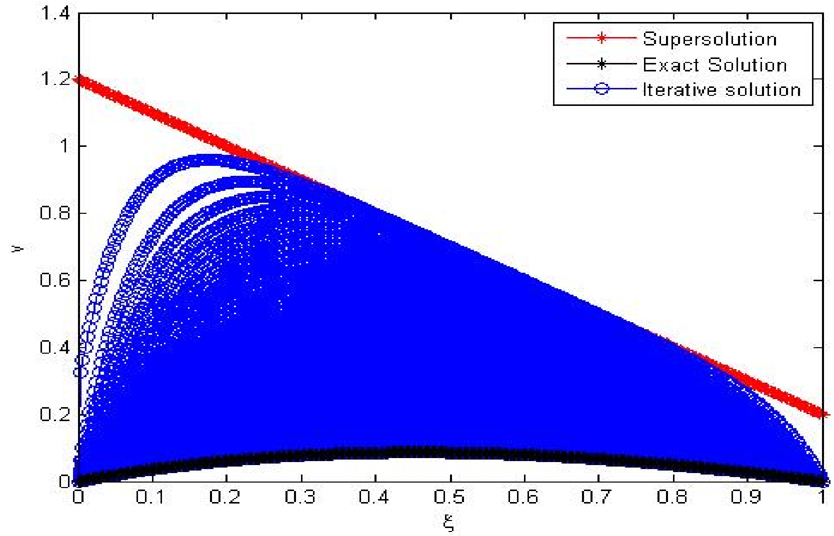
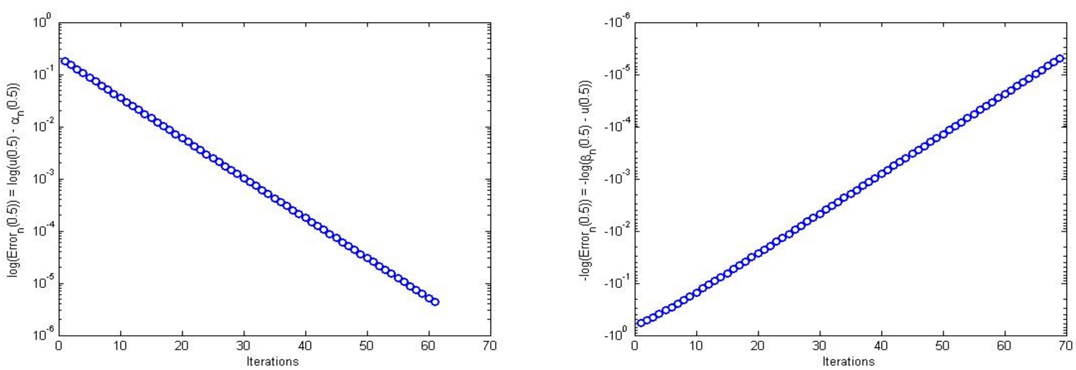


Figure 5.86: Iterations starting from the supersolution for the mesh BVP (5.32).



(a) Iterations starting from subsolution. (b) Iterations starting from supersolution.

Figure 5.87: Monotonicity of the iterates for BVP (1.6).

Figure5.88shows that numerical solution of BVP (5.32) using iterations (4.17) and (4.18) starting from subsolution increases monotonically and converge to true solution.

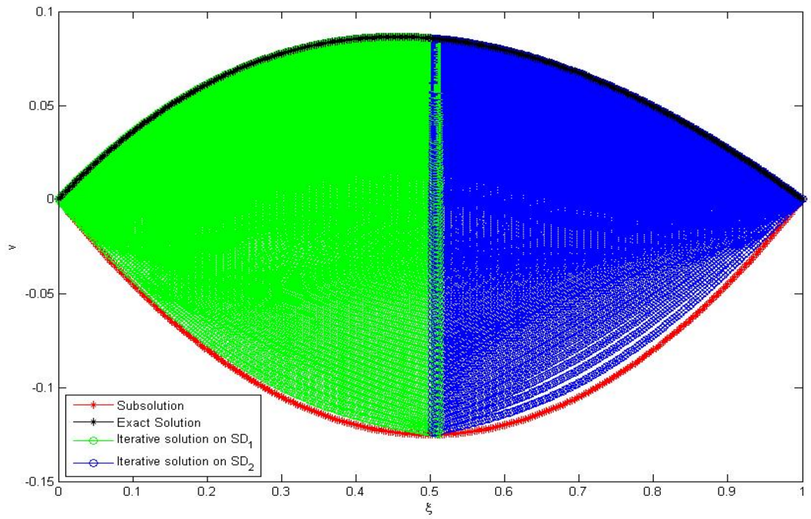


Figure 5.88: Linearized DD iterates starting from subsolution for the mesh BVP (5.32). Similarly Figure5.89shows that numerical solution of BVP (5.32) using iterations (4.27) and (4.28) starting from supersolution decreases monotonically and converge to true solution.

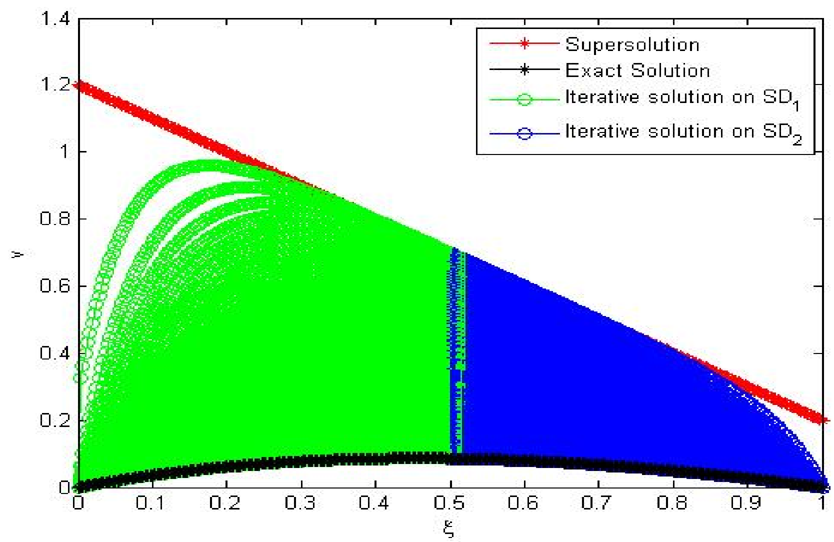
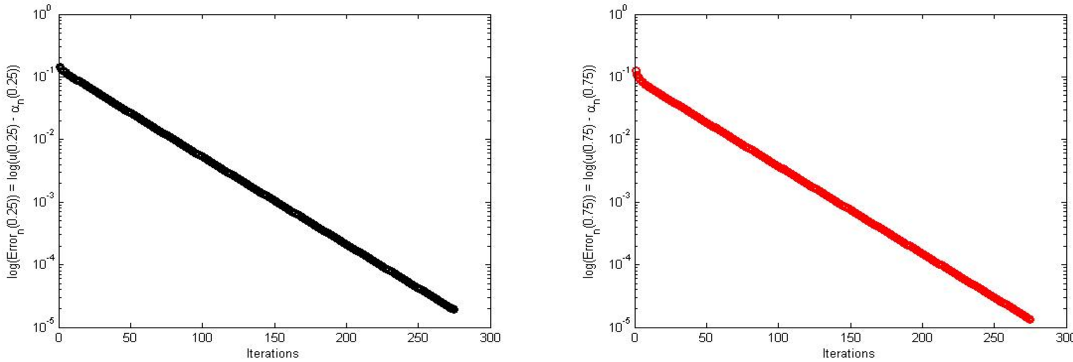


Figure 5.89: Linearized DD iterates starting from supersolution for the mesh BVP (5.32).

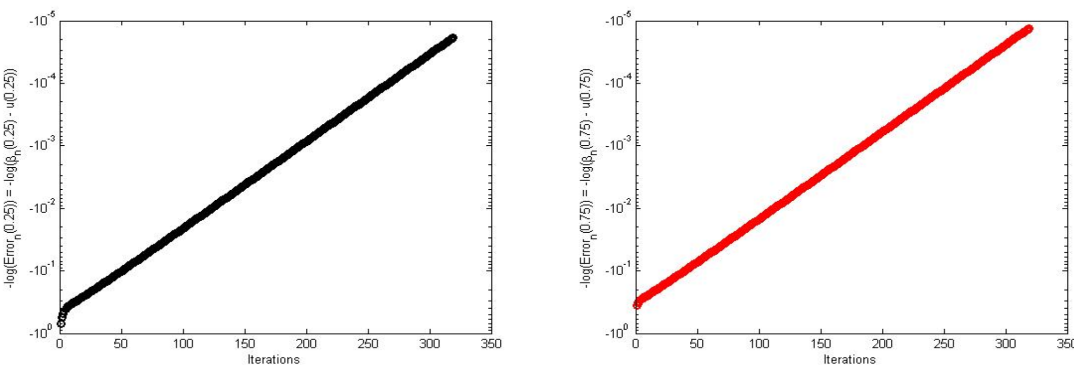
We observe from Figure5.90and5.91that the solution starting from the subsolution increases monotonically and the solution starting from the supersolution decreases monotonically to the true solution.



(a) Monotonicity of iterations on the first subdomain.

(b) Monotonicity of iterations on the second subdomain.

Figure 5.90: Monotonicity of iterates starting from subsolution for the mesh BVP (5.32).



(a) Monotonicity of iterations on the first subdomain.

(b) Monotonicity of iterations on the second subdomain.

Figure 5.91: Monotonicity of iterates starting from supersolution for the mesh BVP (5.32).

Figure5.92shows that inequality (5.25) is true for BVP (5.32), when the iterates start from a subsolution.

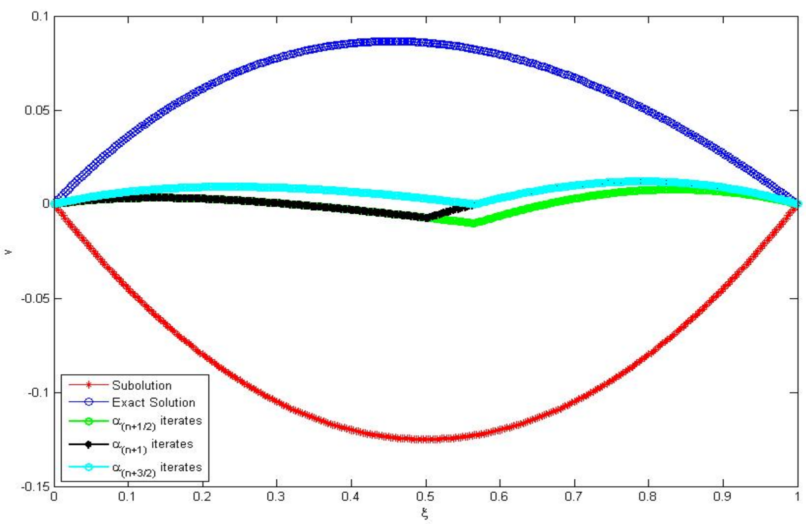


Figure 5.92: Plot showing inequality (5.25) for the mesh BVP (5.32).

Similarly Figure5.93shows that inequality (5.26) is true for BVP (5.32), when the iterates start from a supersolution.

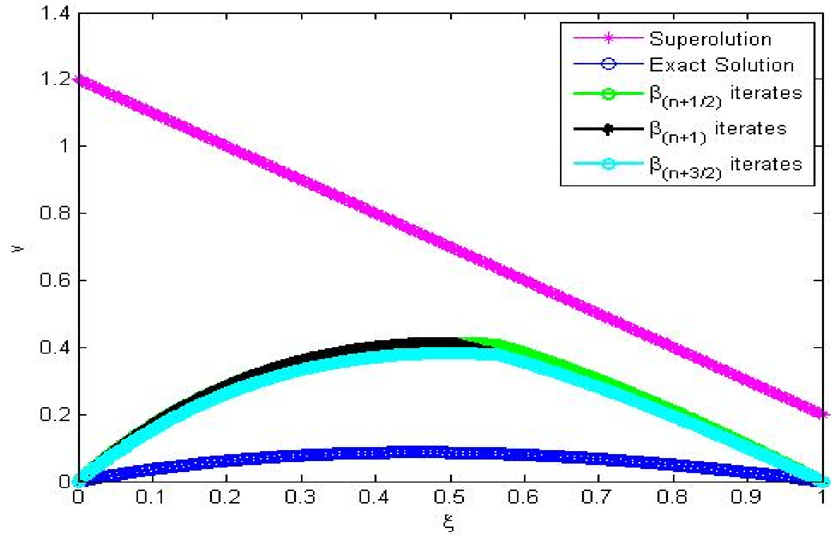


Figure 5.93: Plot showing inequality (5.26) for the mesh BVP (5.32).

When $f(\xi, u, u^{'}) = f(\xi, u)$ then property **[C2]** automatically holds with $N = 0$. Lui [14] assumes that f is Hölder continuous function, this implies f satisfies Lipschitz condition in u. Hence property **[C2]** also holds and this guarantees that we can apply our proposed iteration scheme to obtain the solution. From Example5.3.1.1 and5.3.2.1we observe although our proposed iteration scheme is little bit slower than the Lui's iteration scheme, both these iteration schemes are giving same result.

Chapter 6

Concluding remarks and future work

6.1 Conclusion

The prime focus of this thesis is the linearized domain decomposition solution of BVPs of the form $u'' = f(\xi, u, u')$, where $f(\xi, u, u')$ depends on u' nonlinearly. In this thesis we have successfully extended Cherpion's single domain iterative scheme from [6] to a domain decomposition approach.

Differential equations become large system of equations when they are discretized and we wish to use parallel computers to solve this system. Domain decomposition approaches are suitable to take advantage of parallel computers in order to solve the boundary value problems. In the second chapter, we briefly discussed the existing work of Haynes and Gander [8] on nonlinear domain decomposition methods for the BVPs.

Chapter three presents a single domain linearized scheme for BVPs of the form $u'' = f(\xi, u)$. This scheme is motivated by the domain decomposition scheme analyzed by Lui [14]. We also provide a detailed explanation of Cherpion's linearized single domain iteration to solve the BVPs of the form $u'' = f(\xi, u, u')$ from [6]. This approach requires only the solution of linear system at each iteration.

Chapter four presents an analysis of the linearized domain decomposition method. First the existing work from [14], the linearized DD iterations for problem $u'' = f(\xi, u)$ is discussed. Then our main contribution is presented: a DD extension of Cherpion's iteration for problem of the form $u'' = f(\xi, u, u')$.

Finally numerical results to demonstrate all the theory is provided in Chapter five. In the very beginning of this chapter numerical results for nonlinear domain decomposition approaches is provided. Then the numerical results for the linearized single domain methods is discussed. Lastly some numerical examples of linearized domain decomposition approaches are given. We observed from numerical results that if we increase the overlap, the DD method will give us faster convergence on the other hand if we increase the number of subdomains, the DD method converge more slowly. Theorems to these effects are given in, for example [8] or [20].

6.2 Future research directions

We have extended Cherpion's single domain iterative scheme from [6] to an alternating domain decomposition approach for two subdomains. Next challenge would be to extend the analysis of this scheme to multiple subdomains. We will consider extending the linearized DD scheme to multidimensional problems. Another research challenge is to extend this scheme to an additive or parallel domain decomposition method for two subdomains and several subdomains. Also in the future we will try to develop the theory by relaxing conditions on f for iteration scheme stated in Section4.2, so that we can provide a theoretically sound linearized DD approach to solve mesh BVPs. We will try to relax the global Lipschitz condition on f in u' by using a local Lipschitz condition and Nagumo condition, this has been done for Neumann problems in Cherpion's Thesis [5].

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